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College of energy and environmental sciences

Renewable Energy Department

First Course

Mathematics

Dr. Thamer AL-khafaji

Introduction.

In the name of Allah

Through my teaching in renewable energy department, College of energy and environmental sciences in

AL- Karkh University of science .

I noticed we need to start written book in specializes mathematics in renewable energy.

For this reason I decided write this book for name

The calculus in renewable energy department.

This book contains the first and second course for mathematics in renewable energy.

The chapter one contains the course description for first course such as the tangent and velocity problems, The Limits of function, calculating limits using the limits laws, limits at infinity, continuity of function, Horizontal and vertical asymptote, derivative and Rates of change, the Derivative as a function, differentiation of polynomials, the product and quotient rules, derivative of Trigonometric functions ,Chain Rule, implicit differentiation. Related Rates, Maximum and Minimum values and Mean value theorem, how derivative affect the shape of a Graph. Summary of curve sketching, Optimization problem, ant derivatives, area and derivatives. The define integral, the fundamental theorem of calculus, the indefinite integral and net change theorem, the substitution Rule, areas between curves, volumes by cylindrical shells. Average value of a function, exponential and logarithm functions, derivative and Integrals involving logarithmic functions, inverse functions, derivative and integrals involving Exponential functions, derivative and integrals involving inverse Trigonometric functions. Hyperbolic functions and Hanging cables, indeterminate forms and L′ Hospital ′s Rule.

The chapter two contains the Course Description for second course such as, review of Inverse functions, inverse Trigonometric functions, derivative of inverse Trigonometric functions Hyperbolic functions, Inverse Hyperbolic functions and their derivative, Integrals involving Inverse Trigonometric functions, Integration by parts. Trigonometric integrals, trigonometric substitution, Integrating rational functions by partial fractions, type of improper integrals and method of evaluation, sequences and their limit, monotone sequences, Infinite sequence, the comparison, ratio and Root tests. Alternating series, conditional converges. Maclaurin series and Taylor series, and their approximation power series, differentiating and integrating power series, polar coordinates, curves defined by parametric equations, tangent line and length for parametric and polar curves and area in polar coordinates.

In addition this book contain more than 200 examples solution and more than 100 homework and contain more than 30 illustration forms.

In the end I would like to thank God for help me in completion my book, The mathematics in renewable energy department.

Tanks for all.

For any proposition or any change wrote to my email on the page.

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Chapter one

in this introductory chapter will study Course description for first course written in renewable energy department.

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Contents.

This is Course description for first course wrote in renewable energy contain 14 sections, some times there exist some definitions delay maybe for urgent need.

1- The tangent and velocity problems. The Limits of function.

2- Calculating limits using the limits laws, limits at infinity, Continuity of function.

3-Horizontal and vertical asymptote, derivative and Rates of change.

4- The Derivative as a function, differentiation of polynomials, the product and quotient Rules.

5- Derivative of Trigonometric functions, Chain Rule.

6- Implicit differentiation. Related Rates.

7-Maximum and Minimum values and Mean value theorem.

8-How derivative Affect the shape of a Graph. Summary of curve sketching

9- Optimization problem . Ant derivatives

10- Area and derivatives. The define integral.

The fundamental theorem of calculus.

11- the indefinite integral and net change theorem.

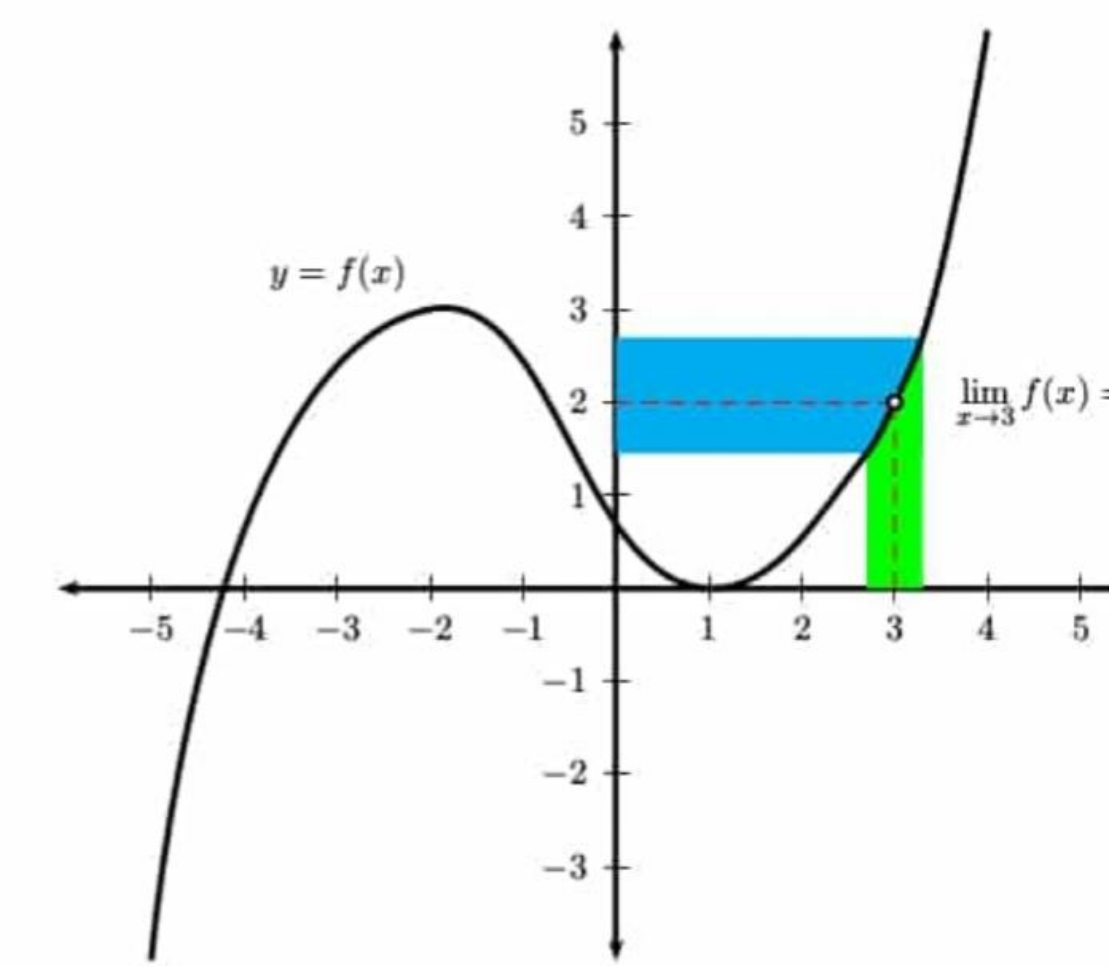
The substitution Rule. Areas between curves.

12- Volumes by cylindrical shells. Average value of a function.

13- Exponential and logarithm functions, Derivative and Integrals involving logarithmic functions.

Inverse functions. Derivative and integrals involving Exponential functions.

14-Derivative and integrals involving inverse Trigonometric functions. Hyperbolic functions and Hanging cables. indeterminate forms and L′ Hospital ′s Rule.


y = f(x)
lim f(x) =
x→3
5
4
3
2
1
−1
−2
−3
−5
−4
−3
−2
−1
1
2
3
4
5

Limits of function.

The concept of a limit is the fundamental building block on which all other calculus concepts are based.

Definition:

$$\lim_{x \to a} f(x) = L$$

Mean that: When a value of f(x) close to (a) the function f(x) approaches the limiting value (L).

Properties:

1- $\lim_{x \to a^+} f(x) = L$

Mean that: (x) approaches (a) from the right.

2- $\lim_{x \to a^-} f(x) = L$

Mean that: (x) approaches (a) from the left.

Note: If $\lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x)$

We say that $\lim_{x \to a} f(x) = L$. Exist, other wise the limit does not exist.

Example:

Find $\lim_{x \to 1} f(x)$, where f(x) = $\begin{cases} x^2 + 1, where & x \leq 1 \\ 3 - x, \ where & x > 1. \end{cases}$

Solution:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 + 1 = 2$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 3 - x = 2.$$

Then

$$\text{Lim}_{x \to 1^+} f(x) = 2 = \lim_{x \to 1^-} f(x)$$

$\therefore \lim_{x \to 1} f(x)$ exist.

Example:

$$find \ \lim_{x \to 0} f(x) \ where \ f(x) = \begin{cases} x^2 + 1, & where \quad x \geq 0 \\ x, & where \quad x < 0 \end{cases}$$

Solution:

$$\text{Lim}_{x \to 0^+} f(x) = \text{Lim}_{x \to 0^+} x^2 + 1 = 1$$

$$\text{Lim}_{x \to 0^-} f(x) = \text{Lim}_{x \to 0^+} x = 0$$

Then

$$\text{Lim}_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$$

$\therefore \lim_{x \to 0} f(x)$ does not exist.

Example:

$$find \ \lim_{x \to -2} f(x) \ where \ f(x) = \begin{cases} x^3 + 1, & where \quad x \geq -2 \\ x - 5, & where \quad x < -2 \end{cases}$$

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Calculating limits using the limits laws.

When we solve the limit, we need to solve by use the laws.

Let : $\lim_{x\to a} f(x) = L_1, \ \lim_{x\to a} g(x) = L_2.$

$1 - \lim_{x\to a}[f(x) \pm g(x)] =$

$\lim_{x\to a} f(x) \pm \ \lim_{x\to a} g(x) = L_1 \pm L_2.$

$2 - \lim_{x\to a}[f(x) * g(x)] =$

$\lim_{x\to a} f(x) * \ \lim_{x\to a} g(x) = L_1 * L_2.$

3- $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ but $\lim_{x\to a} g(x) \neq 0.$

4- $\lim_{x\to a} Kf(x) = K \ \lim_{x\to a} f(x).$

Where K is constant.

5- $\lim_{x\to a} K = K.$

6- $\lim_{x\to a} x = a.$

7- $\lim_{x\to \infty} \frac{1}{x} = 0.$

8- $\lim_{x\to 0^+} \frac{1}{x} = +\infty.$

9- $\lim_{x\to 0^-} \frac{1}{x} = -\infty.$

10- $\lim_{x\to 0} \ \frac{x}{1} = 0.$

11- $\lim_{x\to 0} sinx = 0.$

12- $\lim_{x\to 0} cosx = 1.$

13- $\lim_{x\to 0} \frac{\sin x}{x} = 1.$

14- $\lim_{x\to 0} \frac{x}{sinx} = 1.$

Undefined expression in limits

$$\frac{0}{0}, \frac{\infty}{\infty}, \frac{0}{\infty}, \frac{\infty}{0}, 0 * \infty, \infty * \infty, \infty - \infty$$

But we can say $\infty + \infty = \infty.$

Evaluate the following limits:

Example:

$$\lim_{x\to 1} \frac{x^2 + 1}{x}$$

Solution:

$$\lim_{x\to 1} \frac{x^2 + 1}{x} = \frac{2}{1} = 2.$$

Example:

$$\lim_{x\to -1} \frac{(x+1)(x-1)}{x+1}$$

Solution:

$$\lim_{x\to -1} \frac{(x+1)(x-1)}{x+1} = \lim_{x\to -1} (x-1) = -1 - 1 = -2\ .$$

Example:

$$\lim_{x\to 2} \frac{x^3-8}{x-2}$$

Solution.

$$\lim_{x\to 2} \frac{x^3-8}{x-2} = \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{x-2}$$

$$= \lim_{x\to 2} x^2+2x+4 = 4+4+4 = 12$$

Example:

$$\lim_{x\to 3} \frac{\sqrt{x+1}-2}{(x-3)}$$

Solution.

$$\lim_{x\to 3} \frac{\sqrt{x+1}-2}{(x-3)} \times \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2}$$

$$\lim_{x\to 3} \frac{x+1-4}{(x-3)\left(\sqrt{x+1}+2\right)}$$

$$= \lim_{x\to 3} \frac{(x-3)}{(x-3)\left(\sqrt{x+1}+2\right)}$$

$$= \lim_{x\to 3} \frac{1}{\left(\sqrt{x+1}+2\right)} = \frac{1}{\left(\sqrt{3+1}+2\right)}$$

$$= \frac{1}{2+2} = \frac{1}{4}$$

Example:

$$\lim_{x\to 0}\frac{sin3x}{x}$$

Solution:

$$\lim_{x\to 0}\frac{sin\ 3x}{x}*\frac{3}{3}=\lim_{x\to 0}\frac{3\ \ sin3x}{3x}=3\ \lim_{x\to 0}\frac{sin3x}{3x}=3*1=3$$

limits at infinity.

when a limit approaching to ∞ we can solve the limit by laws, See the example.

Example:

$$\lim_{x\to\infty}\frac{4x^3-2x^2+1}{3x^3-5}.$$

Solution.

$$\lim_{x\to\infty}\frac{4x^3-2x^2+1}{3x^3-5}=\lim_{x\to\infty}\frac{\frac{4x^3}{x^3}-\frac{2x^2}{x^3}+\frac{1}{x^3}}{\frac{3x^3}{x^3}-\frac{5}{x^3}}=\lim_{x\to\infty}\frac{4-\frac{2}{x}+\frac{1}{x^3}}{3-\frac{5}{x^3}}$$

$$=\frac{4}{3}.$$

Example:

$$\lim_{x\to\infty}\frac{x^7-2x^3+10}{x^7+1}.$$

Solution.

$$\lim_{x\to\infty}\frac{x^7-2x^3+10}{x^7+1} = \lim_{x\to\infty}\frac{\frac{x^7}{x^7}-\frac{2x^3}{x^7}+\frac{10}{x^7}}{\frac{x^7}{x^7}+\frac{1}{x^7}} = \lim_{x\to\infty}\frac{1-\frac{2}{x^4}+\frac{10}{x^7}}{1+\frac{1}{x^7}}$$

$= 1.$

Problems:

1- $\lim_{x\to 1}\frac{x^3-1}{x-1}$

2- $\lim_{x\to 0}\frac{sin^2 3x}{x^2}$

3- $\lim_{x\to\infty}\sqrt{x^2+2x}$ -x

4- $\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}$

5- $\lim_{x\to\infty}\frac{x^2-2}{2x^3+3x^2-5}$

6- $\lim_{x\to 1}\frac{(2x+3)(\sqrt{x}-1)}{2x^2+x-3}$

7- $\lim_{x\to 1}\frac{x-1}{\sqrt{x^2+3}-2}$

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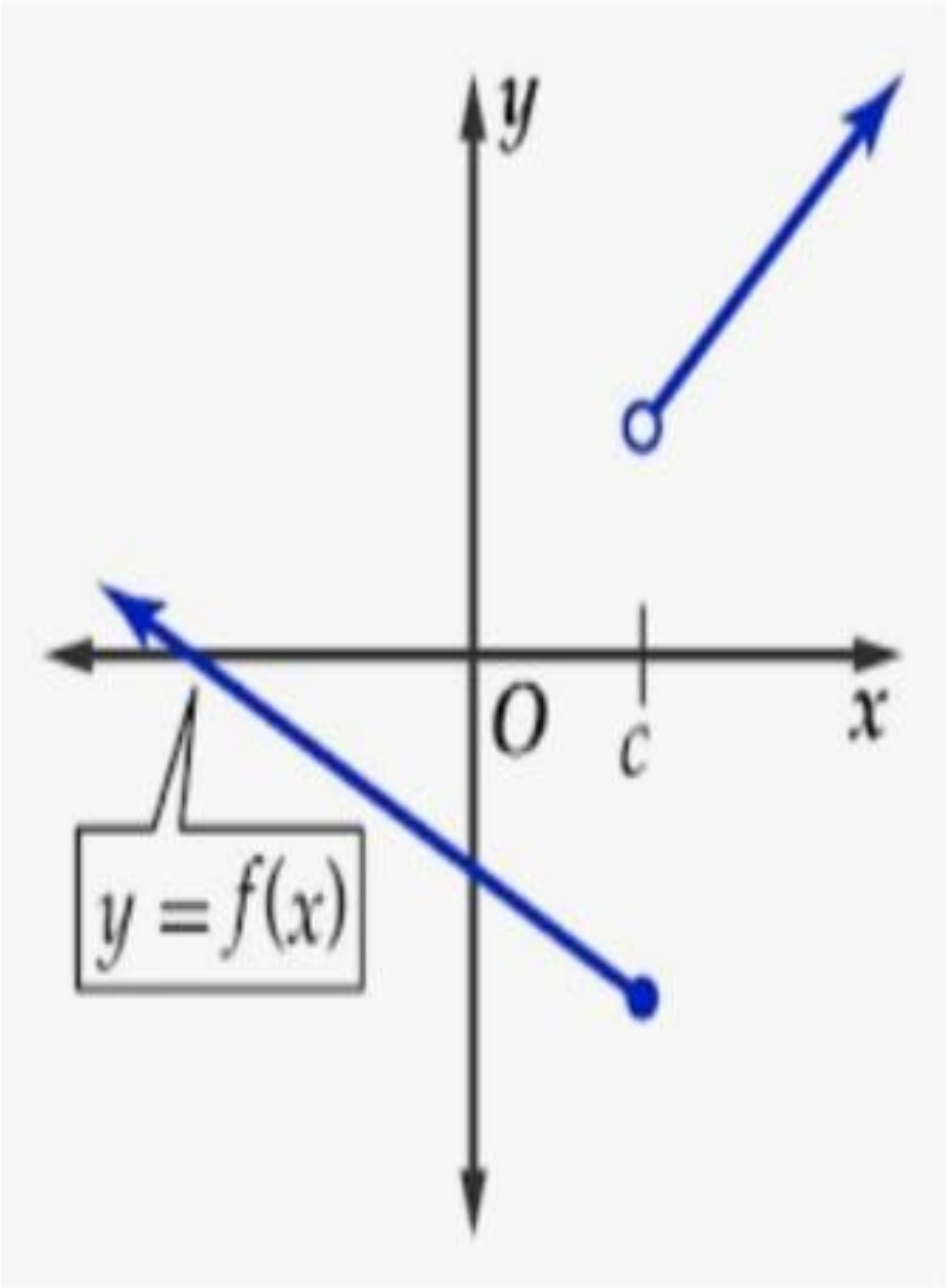

y
O
c
x
y = f(x)

Continuity of function.

Definition:

Continuity of a moving particle on a single path with out unbroken curve ,gaps and jumps or holes such curve can be said to be as continuous.

A function is said to be continuous at x=a if the following conditions are satisfied:

1-The $f(a)$ is exist or defined

2- $\lim_{x\to a} f(x)$ exist

3- $\lim_{x\to a} f(x) = f(a)$.

Otherwise the function is not continuous see the figure

Example:

Check if the function is continuous at x=5, x=0,where

$$f(x) = \begin{cases} x^2 - 1, & where \quad x \geq 5 \\ x, & where \quad x < 5 \end{cases}$$

Solution:

At x=5

1-$f(x) = x^2 - 1 \rightarrow f(5) = 5^2 - 1 = 24.$

2- $\lim_{x\to 5^+} x^2 - 1 = 24.$

$\operatorname{Lim}_{x\to 5^-} x = 5.$

$\therefore \lim_{x \to 5^+} x^2 - 1 \neq \lim_{x \to 5^-} x.$

The limit does not exist, there fore the function is not continuous at x=5.

At x=0. Homework.

Example:

Graph the function $\mathbf{f(x)} = \begin{cases} \boldsymbol{x}, & \boldsymbol{where} \quad \mathbf{0 \leq \boldsymbol{x} \leq 1} \\ \mathbf{2 - \boldsymbol{x}}, & \boldsymbol{where} \quad \mathbf{1 < \boldsymbol{x} \leq 2} \end{cases}$

And if the function is continuous at x=1 ?

Solution:

1- $\boldsymbol{f(x) = x} \to f(1) = 1$

2- $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2 - x = 1$

$\therefore \lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = 1$

3- $\lim_{x \to 1} f(x) = f(1) = 1$

∴ f is continuous at x=1.

Example:

Graph the function $f(x) = \begin{cases} x^2 - 1, & where \quad x \leq 2. \\ 2 - x, & where \quad x > 2. \end{cases}$

And if the function is continuous at x=2 ? Home work.

Example:

Find the value of constant (a) and (b) if the function is

$$f(x) = \begin{cases} x^2 + a & when \quad x \geq 0 \\ 3 + b & when \quad -1 \leq x < 0 \\ x + 5 & when \quad x < -1 \end{cases}$$

when the function is continuous at x=0 and x=-1.

Solution:

At x=0,

1- $f(x) = x^2 + a \rightarrow f(0) = 0^2 + a = a.$

$$2 - \lim_{x \to 0^+} x^2 + a = a$$

$$\lim_{x \to 0^-} 3 + b = 3 + b.$$

Since the function is continuous at x=0, the limit must be exist so

a=3+b ....(1).

At x=-1

1- $f(x) = 3 + b \rightarrow f(-1) = 3 + b = 3 + b$

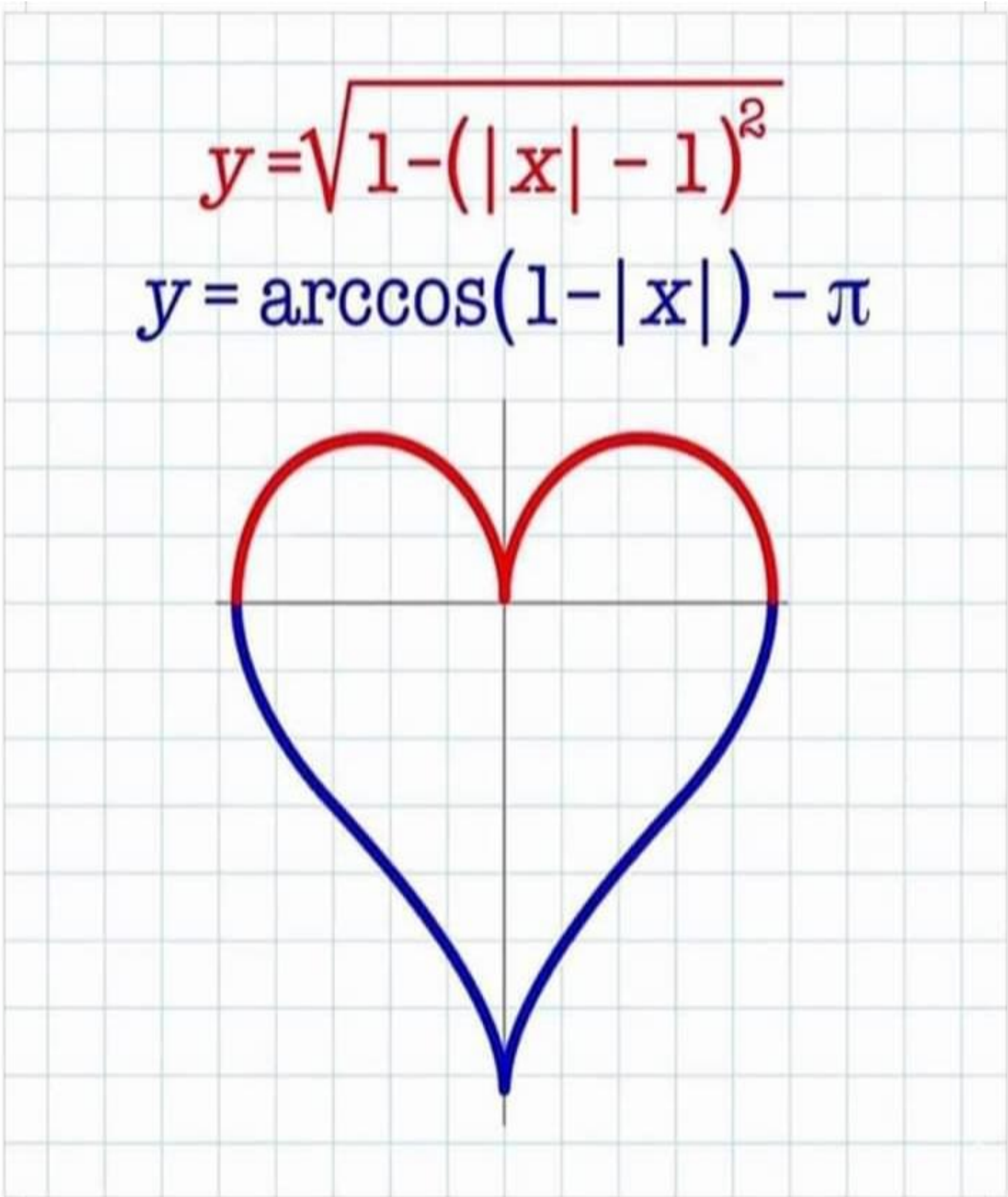
$$2 - \lim_{x \to -1^+} 3 + b = 3 + b$$

$$\lim_{x \to -1^-} x + 5 = 4.$$

Since the function is continuous at x=-1, the limit must be exist so.

4=3+b $\rightarrow b = 1$

By the equation (1) a=4.


$y = \sqrt{1-(|x|-1)^2}$
$y = \arccos(1-|x|) - \pi$

Horizontal and vertical asymptote.

Definition:

A function that can be expressed as a ratio of two polynomials is called a rational function.

If P(x) and Q(x) are polynomials then the domain of the rational function

$$f(x) = \frac{P(x)}{Q(x)},$$

Consists of all values of x such that Q(x)$\neq 0$.

Example:

Find the Horizontal asymptote for the function

$$f(x) = \frac{x^2 + 2x}{x^2 - 1},$$

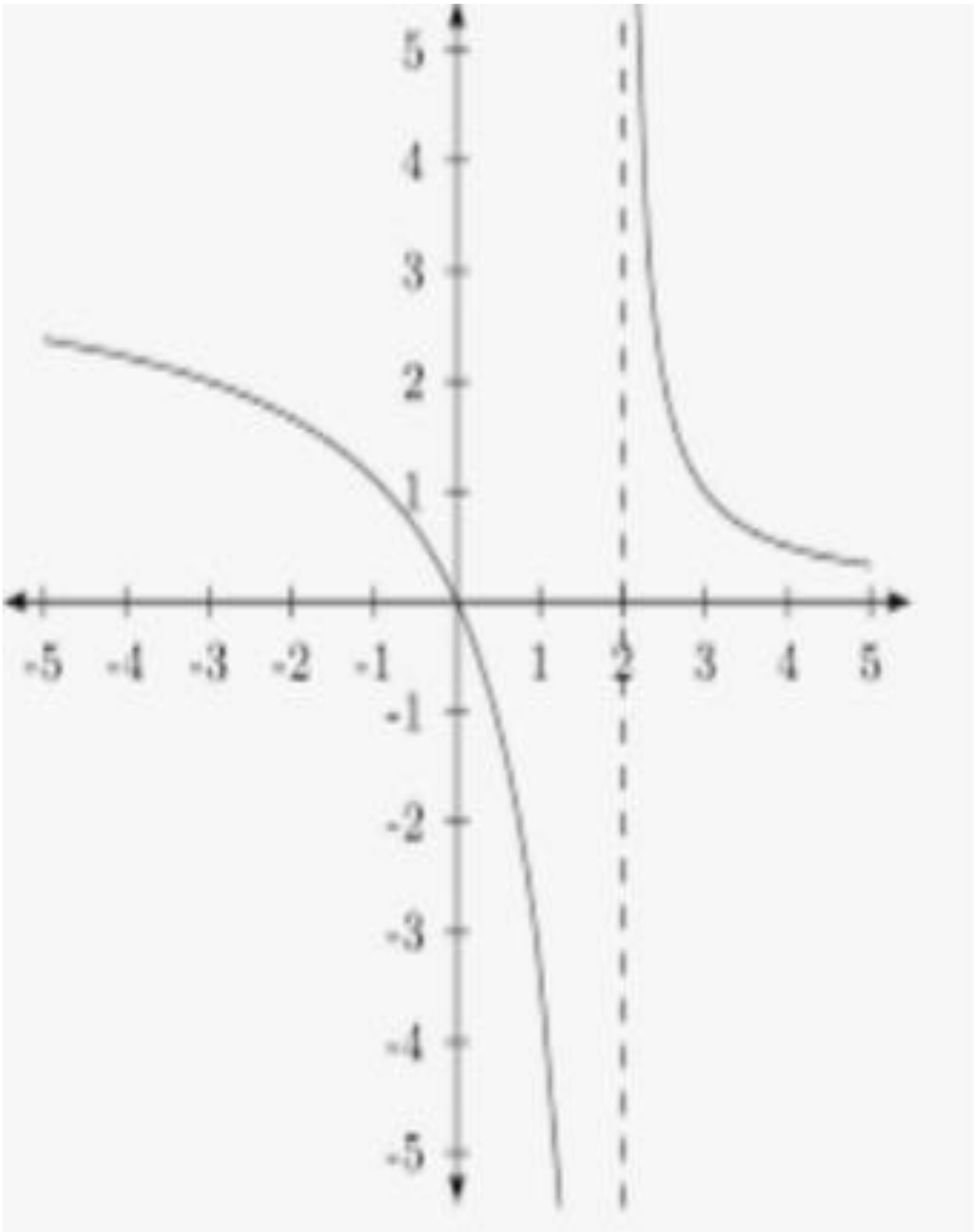
The domain of the rational function consists of all value of x, except x=1 and x=-1.

Vertical Asymptote:

Definition:

Unlike polynomials, rational function may be have numbers at which they are not defined near such points, many (but not all)rational functions have graphs that approximate a vertical line called a Vertical Asymptote.

Horizontal Asymptote:



Definition:

Unlike the graphs of non constant polynomials, which eventually rise or fall indefinitely the graph of many (but not all)rational functions eventually get closer and closer to some horizontal line, called horizontal asymptote.

Now:

The vertical and Horizontal Asymptote for the function.

$$f(x) = \frac{x^2 + 2x}{x^2 - 1},$$

Vertical Asymptote x=1,-1

Horizontal Asymptote y=1.

Example:

Find the Vertical and Horizontal Asymptote for the function

$$f(x) = \frac{3x - 1}{x + 1}$$

Solution:

1-Vertical Asymptote x=-1.

2- Horizontal Asymptote

$$f(x) = y = \frac{3x - 1}{x + 1},$$

$yx + y = 3x - 1$

$yx - 3x = -y - 1$

$x(y - 3) = -y - 1 \quad \div (y - 3)$

$x = \dfrac{-y - 1}{y - 3}$

Horizontal Asymptote y=3.

Example:

Find the Horizontal asymptote of each rational function.

$1 - f(x) = y = \dfrac{5x^3}{x^2 - 4x + 2}$

$2 - f(x) = \dfrac{7x - 2}{x + 3}$

$3 - f(x) = \dfrac{3x^3 - x + 12}{2x^2 - 6x + 7}$

$4 - f(x) = \dfrac{4x + 7}{6x^2 - 5}$

$5 - f(x) = \dfrac{8x^2 - 5x + 1}{4x^2 - 3}$

Answers.

1) None

2) y = 7

3) $y = \frac{3}{2}$

4) y = 0

5) y = 2

Problems:

Find the Vertical and Horizontal Asymptote for the function and graph.

$1 - \ f(x) = \dfrac{x^2}{x^2+1}$

$2 - f(x) = \dfrac{x-1}{x+1}$

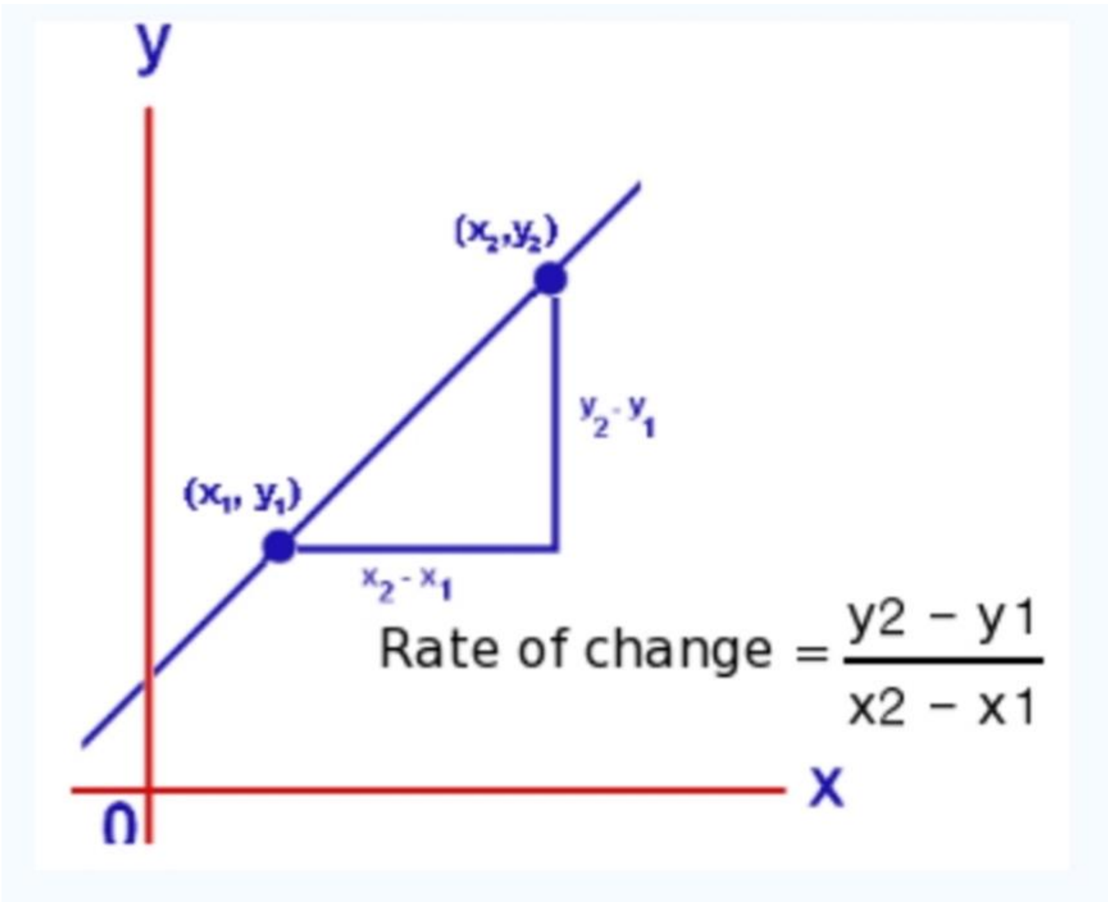
$3 - y = \dfrac{x^2-1}{x^2+1}$

$4 - f(x) = \dfrac{6}{x^2+3}$

$5 - y = \dfrac{x^2-1}{x^2-2x-3}$

$6 - f(x) = \dfrac{3}{x^2+1}$

$7 - f(x) = x^6 - x^2 + 3x$



y
(x_2, y_2)
$y_2 - y_1$
(x_1, y_1)
$x_2 - x_1$
Rate of change = $\frac{y2 - y1}{x2 - x1}$
x
0

Derivative and Rates of change

Definition:

If y=f(x), then the average rate of change of y with respect to x over the interval $[x_0, x_1]$ is.

$$r_{ave} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Geometrically:

The average rate of change of y with respect to x over interval $[x_0, x_1]$ is the slope of the secant line to the graph of y=f(x) through the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$.

Definition:

If y=f(x) then the instantaneous rate of change of y with respect to x when x= x_0

$$r_{inst} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Example:

Find the average rate of change of y with respect to x over the interval [3,5] , y=x^2+1.

Solution:

$$r_{ave} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(5) - f(3)}{5 - 3} = \frac{26 - 10}{2} = 8$$

Thus, on the average, y increases 8 unit per unit in crease in x over the interval [3,5].

Example:

Find the average rate of change of y with respect to x over the interval [0,180], y=cos x.

Solution:

$$r_{ave} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(180) - f(0)}{180 - 0}$$

$$= \frac{cos180 - \cos 0}{180 - 0} = \frac{-2}{180} = \frac{-1}{90}$$

Thus, on the average, y decreases $\frac{-1}{90}$ unit per unit in crease in x over the interval [0,180].

Example:

Find the instantaneous rate of change of y with respect to x when x=-4, y=$x^2 + 1$

Solution:

$$r_{inst} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$r_{inst} = \lim_{x_1 \to -4} \frac{x^2 + 1 - 17}{x + 4} = \lim_{x_1 \to -4} \frac{x^2 - 16}{x + 4}$$

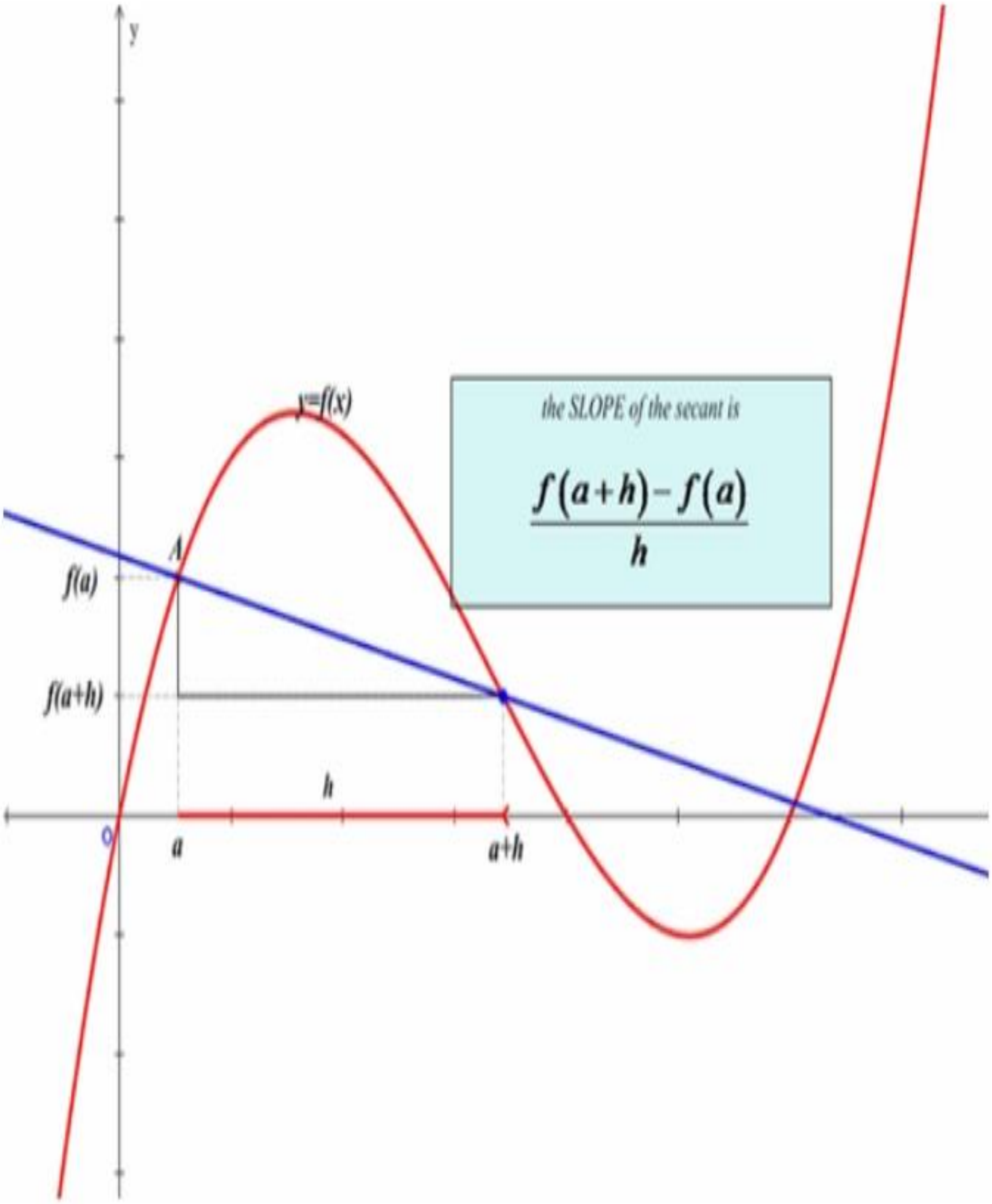
$$r_{inst} = \lim_{x_1 \to -4} \frac{(x + 4)(x - 4)}{x + 4}$$

$$r_{inst} = \lim_{x_1 \to -4} x - 4 = -8.$$

For small change in x from x=-4, the value of y will change approximately eight times as much in the negative direction.

<u>Exercise:</u>

Find the instantaneous rate of change of y with respect to x at the general point corresponding to x=x_0, where y=x^2+1.


y
y=f(x)
the SLOPE of the secant is
$\frac{f(a+h)-f(a)}{h}$
A
f(a)
f(a+h)
h
0
a
a+h

Derivative as a function.

Definition:

Let $y = f(x)$ define a function of x if

$\lim_{\Delta x \to 0} \quad \frac{f(x+\Delta x)-f(x)}{\Delta x}$ exist and finite, we call this limit the derivative of f at x and denoted by

$f'(x), \frac{df(x)}{d(x)}, \frac{dy}{dx}, y'$ and say that f is differentiable at x.

Remark:

1-Let y=f(x), y=$\Delta y = f(x + \Delta x)$ then

$\Delta y = f(x + \Delta x) - y$

$\Delta y = f(x + \Delta x) - f(x)$

$2 - \ \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$

3- if $f(x + \Delta x) - f(x) = \Delta y$ then $\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$

$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}.$

Example:

Let y=f(x)= c, use the definition of derivative of limit to find y′

Solution:

The definition of derivative is

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{c - c}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0.$$

We get if f(x)=c then f′(x)=0.

Example:

Let y=f(x)=x^2, use the definition of derivative of limit to find y′

Solution:

The definition of derivative is

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} 2x + \Delta x = 2x.$$

We get if f(x)=x^2 then f′(x)=2x.

Example:

Let $y = f(x) = \sqrt{x}$, use the definition of derivative of limit to find y′

Solution:

The definition of derivative is

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x}$$

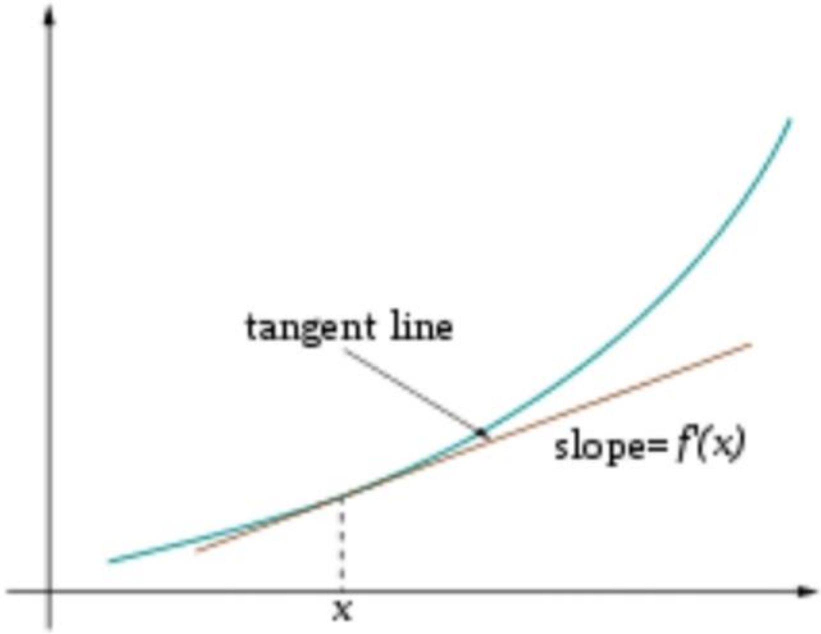
$$f'(x) = \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} * \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{1}{(\sqrt{x + \Delta x} + \sqrt{x})} = \frac{1}{\sqrt{x} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

We get if f(x)=$\sqrt{x}$ then f′(x)=$\frac{1}{2\sqrt{x}}$



The tangent and velocity problems.

Remark:

1- The derivative of a function f at the point x=a is the slope of the tangent line to the curve of f at the point (a, f(a)).

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}.$$

2- $y - y_0 = m\,(x - x_0)$ The equation of line tangent

3- $y - y_0 = \frac{-1}{m}(x - x_0)$ The equation of perpendicular line

Example:

Find the equation of the line tangent and the equation of perpendicular line to the curve $y = \sqrt{x}$ at x=1.

Solution:

Since x=1 then $f(x) = y = \sqrt{x} = \sqrt{1} = 1,$

then the point (1,1) is called tangent point,

The slope m=Derivative of f at the point (1,1).

$$y = \sqrt{x} = (x)^{\frac{1}{2}}$$

$$m = y' = \frac{dy}{dx} = \frac{1}{2}(x)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

The slope at the point $(1,1) = \frac{dy}{dx}_{(1,1)} = \frac{1}{2\sqrt{x}}_{(1,1)} = \frac{1}{2}$

$y - y_0 = m(x - x_0)$ The equation of line tangent

$$y - 1 = \frac{1}{2}(x - 1) \quad * 2$$

$$2y - 2 = x - 1$$

$2y - x - 1 = 0.$ The equation of line tangent

perpendicular line $y' = -2$

$$y - y_0 = \frac{-1}{m}(x - x_0)$$

$$y - 1 = -2(x - 1)$$

$y - 1 = -2x + 2$

$y + 2x - 3 = 0.$ The equation of perpendicular line.

Example:

Find the equation of the line tangent and the equation of perpendicular line to the curve $y = x$ at (2,5) by the definition of derivative.

Solution:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{\Delta x} = 1$$

The slope m=1

$y - y_0 = m(x - x_0)$ The equation of line tangent

$y - 5 = 1(x - 2)$

$y - x - 3 = 0.$ The equation of line tangent

Equation of perpendicular line (Exc)

Problems:

Find the equation of the line tangent and the equation of perpendicular line to the curve $y = \sqrt{x} - 10$ at x=4.

Solution.

$y = \sqrt{4} - 10 = -8$

$$\frac{dy}{dx}_{(4,-8)} = \frac{1}{2\sqrt{x}}_{(4,-8)} = \frac{1}{4}$$

The slope.

$y - y_0 = m(x - x_0)$ the equation of line tangent

$$y + 8 = \frac{1}{4}(x - 4)$$

$4y + 23 = x - 4.$

$4y - x + 27 = 0$

The equation of perpendicular line (Exc)

Theorem:

If f is differentiable at x_0 then f is continuous at x_0.

Remark:

The converse of above theorem is not true since the function may be continuous at x_0, but not differentiable at x_0.

The function $y = |x|$ is continuous at x=0

but not differentiable at x=0.

Differentiation of polynomials.

In this section we will develop some important theorems that will enable us to calculate derivatives more efficiently.

Theorem:

$1 - \dfrac{d}{dx}\, c = 0.$

$2 - \dfrac{d}{dx}\, x^n = nx^{n-1}.$

$3 - \dfrac{d}{dx}\, c\, f(x) = c\, f'(x).$

$4 - \dfrac{d}{dx}\big(u(x) \pm Q(x)\big) = u'(x) \pm Q'(x).$

$5 - \dfrac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} f'(x).$

$6 - \dfrac{d}{dx}\sqrt[n]{f(x)} = \dfrac{f'(x)}{n\sqrt[n]{f}\ \ (x)}.$

Example:

$let\ y = 100.\ Find\ \frac{dy}{dx}$

Solution:

$$\frac{dy}{dx} = 0.$$

Example:

$let\ y = 7x^2 + 12x + \sqrt{3}.\ \ Find\ \frac{dy}{dx}$

Solution:

$$\frac{dy}{dx} = 14x + 12.$$

Example:

$let\ y = \frac{x^3}{3} + \frac{x^2}{2} + x - 1.\ Find\ \frac{dy}{dx}$

Solution:

$$\frac{dy}{dx} = \frac{3x^2}{3} + \frac{2x}{2} + 1 = x^2 + x + 1.$$

Example:

$let\ y = (x^4 + 5)^4.\ Find\ \frac{dy}{dx}$

Solution:

$$\frac{dy}{dx} = 4(x^4 + 5)^3\ \ 4x^3 = 16x^3(x^4 + 5)^3.$$

$$Find\ \frac{ds}{dt}\ of\ the\ following\ functions.$$

$$S = (t + t^{-1})^2$$

Solution:

$$\frac{ds}{dt} = 2(t + t^{-1})^1(1 - t^{-2}) = (2t + 2t^{-1})(1 - t^{-2})$$

Exercise:

$$Find\ \frac{dy}{dx}\ of\ the\ following\ functions.$$

$$1 - y = (x^4 + 2x^3 - 25)^{\frac{2}{3}}$$

$$2 - y = \sqrt{x^3 + 12x^2 + 30}$$

Problems:

$$Find\ \frac{ds}{dt}\ of\ the\ following\ functions.$$

$$1 - S = \left(t \quad + t^{-1}\right)^3.$$

$$2 - S = \sqrt[3]{t^2 + 7t + 12}.$$

Product and quotient rules.

$$1 - \ \frac{d}{dx}\left(u(x).Q(x)\right) = u(x)Q'(x) + Q(x)u'(x).$$

$$2 - \ \frac{d}{dx}\left(\frac{u(x)}{Q(x)}\right) = \frac{Q(x)\,u'(x) - u(x)Q'(x).}{(Q(x))^2}$$

Example:

$$\text{Let } y = (x^4 - 2x + 100)(6x^3 - 15x^2 + 33). \text{ Find } \frac{dy}{dx}$$

Solution.

By $\frac{d}{dx}\big(u(x).Q(x)\big) = u(x)Q'(x) + Q(x)u'(x)$.

$$\frac{dy}{dx} = (x^4 - 2x + 100)\,(18x^2 - 30x) + (6x^3 - 15x^2 + 33)(4x^3 - 2).$$

Example:

$$let\ y = \frac{x^3 - 5}{x^2 + 2x + 10}.\ Find\ \frac{dy}{dx}$$

Solution:

$$By\quad \frac{d}{dx}\left(\frac{u(x)}{Q(x)}\right) = \frac{Q(x)\,u'(x) - u(x)Q'(x).}{(Q(x))^2}$$

$$\frac{dy}{dx} = \frac{(x^2 + 2x + 10)3x^2 - (x^3 - 5)(2x + 2).}{(x^2 + 2x + 10)^2}$$

Example:

$$let\ y = (x\quad + 1)^2\ (x^2 + 4)^{-3}.\ Find\ \frac{dy}{dx}$$

Solution:

$$\frac{dy}{dx} = (x\quad + 1)^2\,(-3(x^2 + 4)^{-4}\ 2x) + \ (x^2 + 4)^{-3}(2(x + 1))$$

$$\frac{dy}{dx} = (x \quad + 1)^2 \left(-6x(x^2+4)^{-4}\ \right)$$
$$+ \quad (x^2+4)^{-3}(2(x+1))$$

$$\frac{dy}{dx} = \frac{-6x(x \quad + 1)^2}{(x^2+4)^4} + \frac{2(x+1)}{(x^2+4)^3}.$$

Example:

$$let\ y = \left(\frac{x+1}{x-1}\right)^2\ Find\ \frac{dy}{dx}$$

Solution:

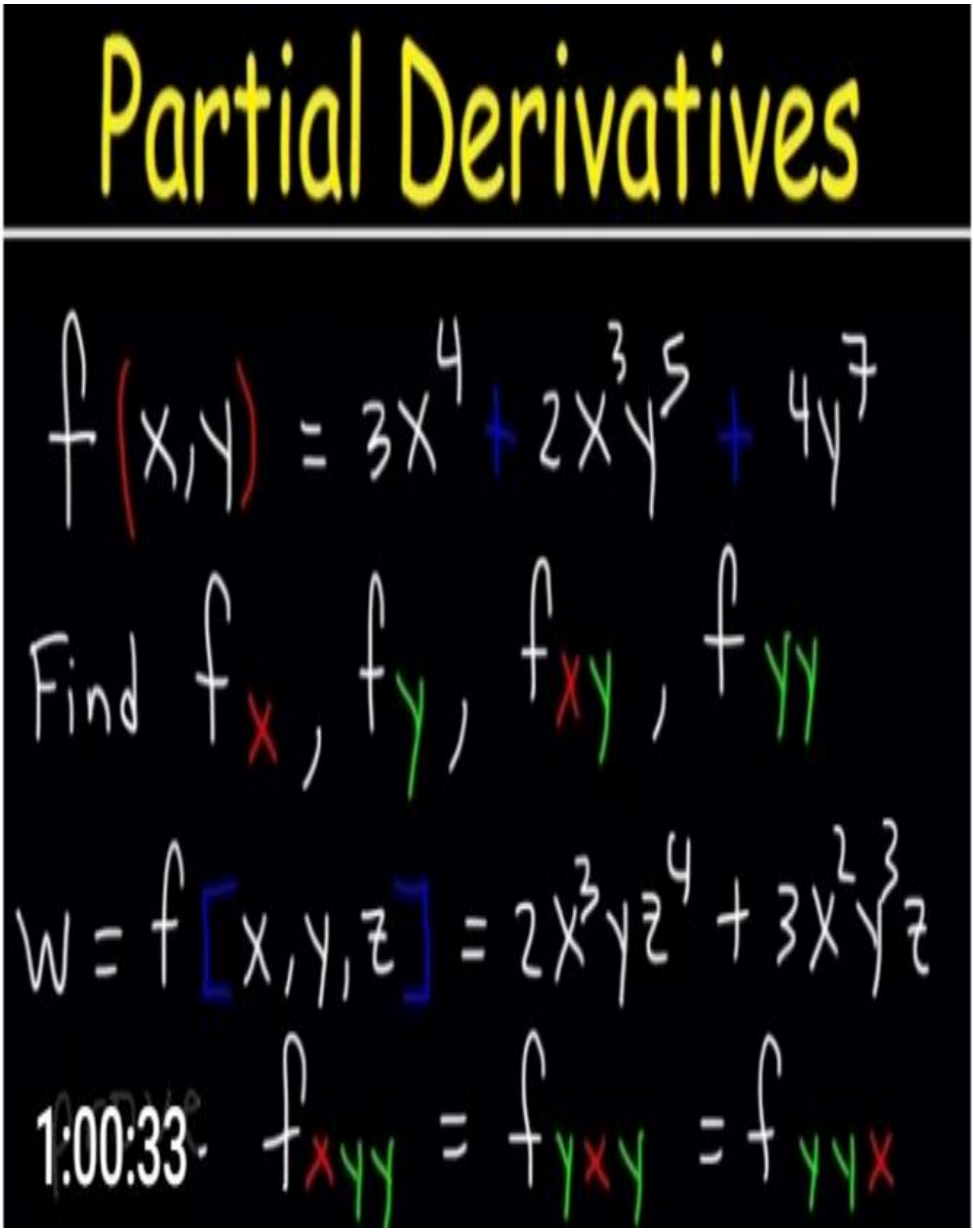
$$\frac{dy}{dx} = 2\left(\frac{x+1}{x-1}\right)\ \left(\frac{(x-1)-(x+1)}{(x-1)^2}\right)$$

Problems:

$$Find\ \frac{ds}{dt}\ of\ the\ following\ functions.$$

$$1 - S = \frac{1}{t^2+1}.$$

$$2 - S = \frac{2t}{3t^2+1}.$$


Partial Derivatives
f(x,y) = 3x^4 + 2x^3y^5 + 4y^7
Find f_x, f_y, f_xy, f_yy
W = f[x,y,z] = 2x^3yz^4 + 3x^2y^3z
1:00:33
f_xyy = f_yxy = f_yyx

Implicit differentiation.

Let $z = f(x, y)$ be a function of two independent variable x and y.

1- If y is fixed then f will be a function of one variable than we can derive with respect to (w.r.t) x. this derivative is called partial derivative of f (w.r.t) x and denoted by $\frac{\partial f}{\partial x}$, hence f_x is a function and its value at (x_0, y_0).
2- If x is fixed then f will be a function of one variable than we can derive with respect to (w.r.t) y. this derivative is called partial derivative of f (w.r.t) y and denoted by $\frac{\partial f}{\partial y}$, hence f_y is a function and its value at (x_0, y_0).

Example:

Find f_x, f_y of the following function.

$$f(x, y) = 5xy - 7x^2 - y^2 - 3x - 6y - 2$$

Solution:

$$f_x = 5y - 14x - 3$$

$$f_y = 5x - 2y - 6$$

Example:

Find f_x, f_y of the following function

$$f(x, y) = \sqrt{9 - x^2 - y^2} = (9 - x^2 - y^2)^{\frac{1}{2}}$$

Solution:

$$f_x = \frac{1}{2}(9 - x^2 - y^2)^{\frac{-1}{2}} \ (-2x) = \frac{-x}{\sqrt{9 - x^2 - y^2}}$$

$$f_y = \frac{1}{2}(9 - x^2 - y^2)^{\frac{-1}{2}} \ (-2y) = \frac{-y}{\sqrt{9 - x^2 - y^2}}$$

Exercise:

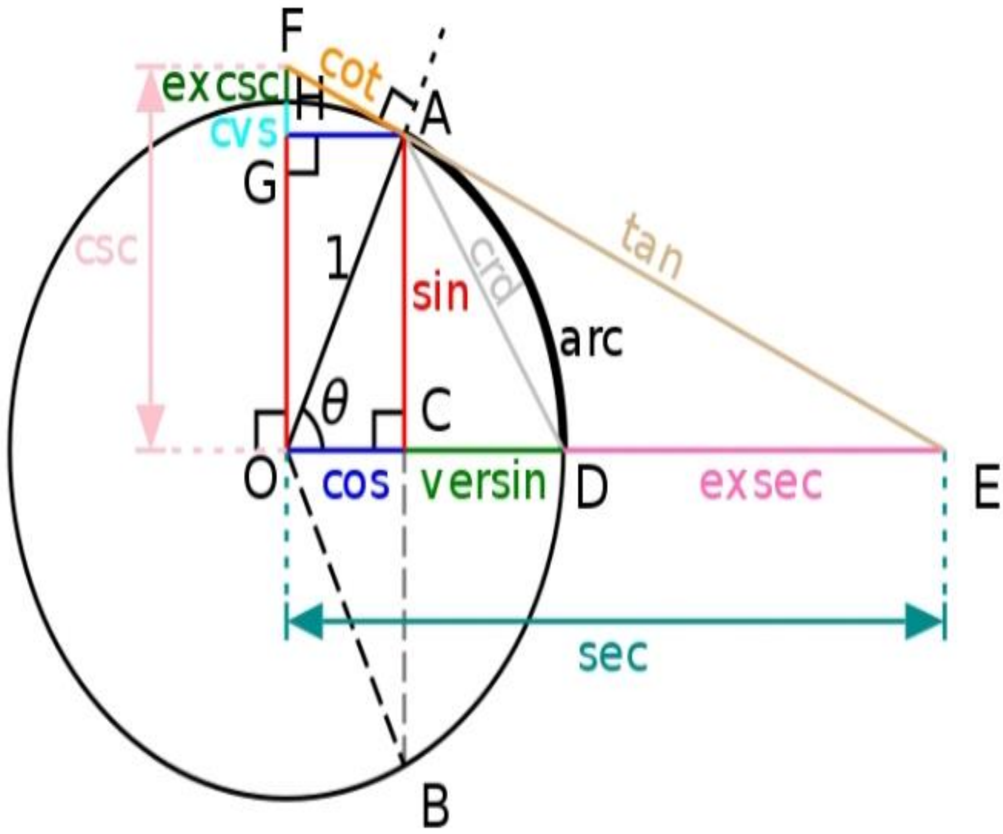
Find f_x, f_y of the following functions

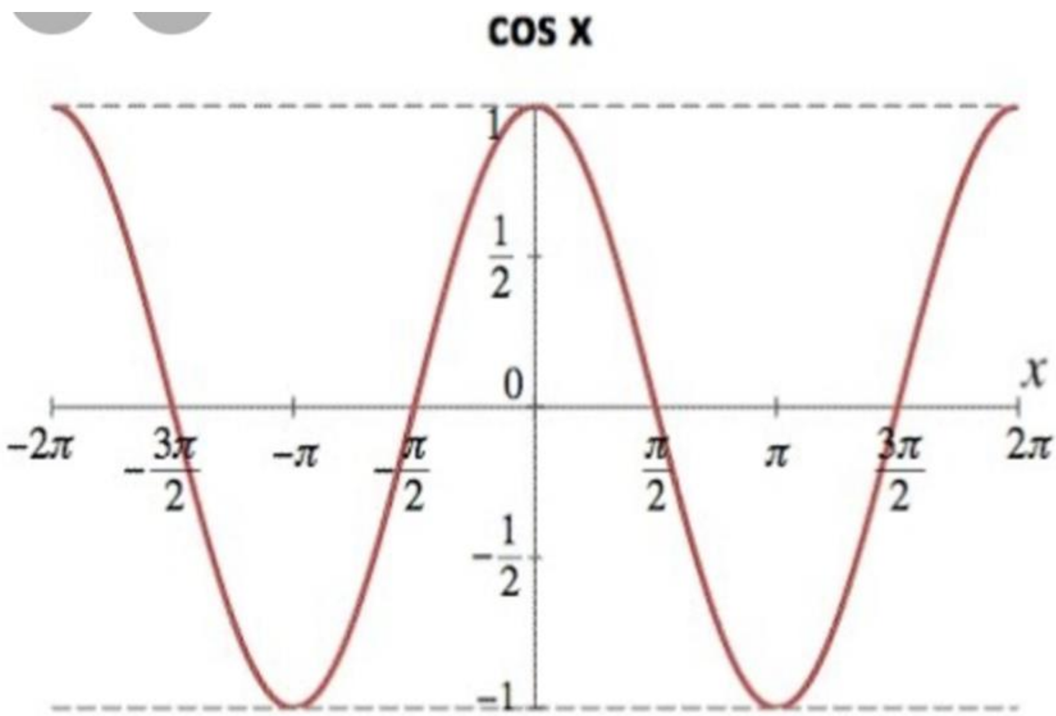
$1 - f(x, y) = e^{xy^2}$

$2 - f(x, y) = tanx\ siny$

$3 - f(x, y) = \frac{1}{xy}$

$4 - f(x, y) = \ln(x - 2xy + y^2)$


F
excsc
H
cot
A
cvs
G
csc
1
sin
crd
tan
arc
θ
C
O
cos
versin
D
exsec
E
sec
B


cos x
1
1/2
0
x
-2π
-3π/2
-π
-π/2
π/2
π
3π/2
2π
-1/2
-1

Derivative of Trigonometric functions.

Theorem.

$$1-\frac{d}{dx}\ sinu = cosu\ \frac{du}{dx}$$

$$2-\frac{d}{dx}\ cosu = -sinu\frac{du}{dx}$$

$$3-\frac{d}{dx}\ tanu = Sec^2u\frac{du}{dx}$$

$$4-\frac{d}{dx}\ cotu = -cSc^2u\frac{du}{dx}$$

$$5-\frac{d}{dx}\ Secu = Secu\ tanu\ \frac{du}{dx}$$

$$6-\frac{d}{dx}\ cScu = -cScu\ cotu\ \frac{du}{dx}$$

Example:

$$Find\ \frac{dy}{dx}\ of\ the\ following\ functions.$$

$$y = cos10x$$

Solution:

$$\frac{dy}{dx} = -sin10x.10 = -10sin10x.$$

Example:

$$Find\ \frac{dy}{dx}\ if\ the\ function\ y = sinx^4$$

Solution:

$$\frac{dy}{dx} = 4x^3 cosx^4.$$

Example:

$$Find\ \frac{dy}{dx}\ if\ the\ function\ y = cos\sqrt{12x}$$

Solution:

$$\sqrt{12x} = (12x)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}(12x)^{\frac{-1}{2}}\ 12 = 6(12x)^{\frac{-1}{2}} = \frac{6}{\sqrt{12x}}$$

$$\frac{dy}{dx} = \frac{-6\ sin\sqrt{12x}}{\sqrt{12x}}$$

Example:

$$Find\ \frac{dy}{dx}\ if\ the\ function\ y = sin^4 3x$$

Solution:

$$y = sin^4 3x = (\sin 3x)^4$$

$$\frac{dy}{dx} = 4\,(\sin 3x)^3\ cos3x\ 3 = 12\,cos3x\ (\sin 3x)^3$$

Example:

$$Find\ \frac{dy}{dx}\ if\ the\ function\ y = \tan(-4x)$$

Solution:

$$\frac{dy}{dx} = sec^2(-4x).(-4) = -4sec^2(-4x).$$

Example:

$$Find\ \frac{dy}{dx}\ if\ the\ function\ y = \mathrm{Sec}^2 4x$$

Solution:

$$y = \mathrm{Sec}^2 4x = (\mathrm{Sec}\,4x)^2$$

$$\frac{dy}{dx} = 2\ (Sec4x)\ (Sec4x.tan4x)4 = 8\ \mathrm{Sec}^2 4x\ tan4x.$$

Example:

$$Find\ \frac{dy}{dx}\ if\ the\ function\ \ y = \mathrm{Cot}\left(\frac{x}{x+1}\right)$$

Solution:

$$\frac{dy}{dx} = -cSc^2\left(\frac{x}{x+1}\right).\frac{(x+1)-x.}{(x+1)^2} = \frac{-1}{(x+1)^2}cSc^2\left(\frac{x}{x+1}\right).$$

Example:

$$Find\ \frac{dy}{dx}\ .y = \mathrm{Sec}(1-\mathrm{x})$$

Solution:

$$\frac{dy}{dx} = \mathrm{Sec}(1 - \mathrm{x})\tan(1 - \mathrm{x})(-1)$$

$$= -\mathrm{Sec}(1 - \mathrm{x})\ \tan(1 - \mathrm{x}).$$

Example:

$Find\ \frac{dy}{dx}\ .\, y = \sin^2 x^4$

Solution:

$$y = \sin^2 x^4 = (\sin\ \ x^4)^{\mathbf{2}}$$

$$\frac{dy}{dx} = 2\,(\sin\ \ x^4)\cos x^4\ 4x^3 = 8x^3 \sin\ \ x^4 \cos x^4.$$

Example:

$Find\ \frac{dy}{dx}$ .If $y = \mathrm{sinx\ \ cosx}$

Solution:

$$\frac{dy}{dx} = \sin x\,(-sinx) + cosx\ cosx = cos^2x - sin^2x.$$

Example:

$Find\ \frac{dy}{dx}\,.\,if\ \ y = \cos\sqrt{\mathrm{x}}$

Solution:

$$\frac{dy}{dx} = -sin\sqrt{x}\quad \frac{1}{2\sqrt{x}}.$$

Example:

$Find\ \frac{dy}{dx}.\ if\ y = \tan x$

Solution:

$$y = \tan x = \frac{sinx}{cosx}$$

$$\frac{dy}{dx} = \frac{cosx\ cosx - [-sinx\ sinx]}{cos^2x} = \frac{cos^2x + sin^2x}{cos^2x} = \frac{1}{cos^2x} = sec^2x$$

Example:

$$Find\ \frac{dy}{dx} \qquad .x = \tan y$$

Solution:

$$1 = sec^2y.\frac{dy}{dx} \quad \div sec^2y$$

$$\frac{dy}{dx} = \frac{1}{sec^2y} = \frac{1}{1 + tan^2y} = \frac{1}{1 + x^2}.$$

Exercise:

$$\boldsymbol{Find\ \frac{dy}{dx}\ of\ the\ following\ Trigonometricfunctions}.$$

$1 - y = \cot x.$

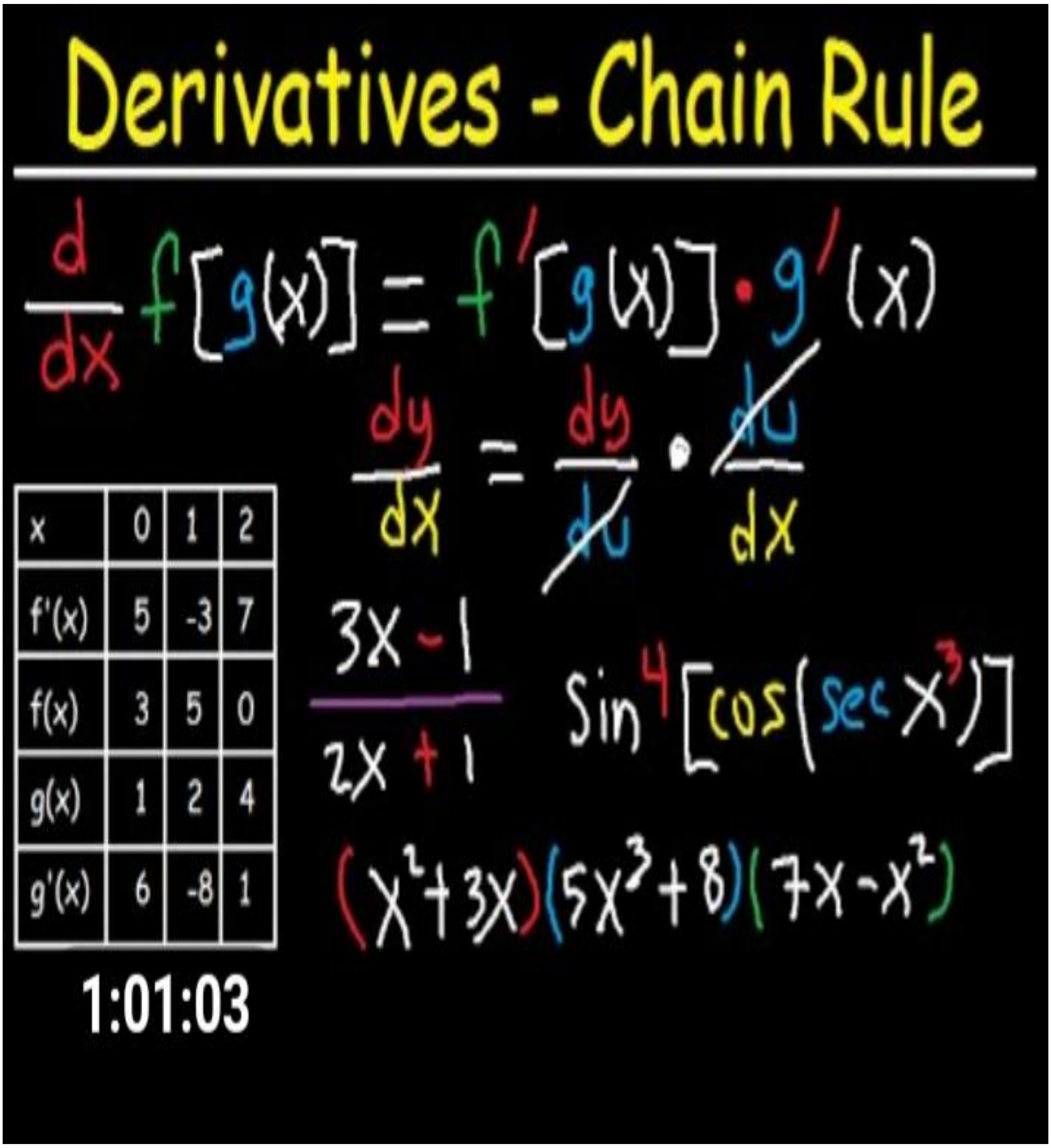
$2 - y = [\cot(x^2 + 3x - 5]^3$

$3 - y = \sec x = \frac{1}{cosx}$

$4 - y = sec^2x - tan^2x.$

$5 - y = \text{cscx} = \frac{1}{sinx}.$

$7 - y = \text{secx}(\text{csc}(\alpha x + 1)).$



Chain Rule.

Definition:

If g is differentiable at x and f is differentiable at g(x) then the composition f(g(x)) is differentiable at x.

If

$$y = f(t), \qquad t = g(x) \quad then \quad \frac{dy}{dx} = \frac{dy}{dt}\ \frac{dt}{dx}.$$

$$\text{i.e}\ \ \frac{d}{dx}[\mathrm{f(g(x)]} = f'\big(g(x)\big).g'(x).$$

Example:

$$Find\ \frac{dy}{dx}\ if\ \ y = cost \quad and\ \ t = x^2 + 10x + 12.$$

Solution:

$$\frac{dy}{dx} = \frac{dy}{dt}\ \frac{dt}{dx}$$

$$\frac{dy}{dt} = -sint$$

$$\frac{dt}{dx} = 2x + 10$$

$$\frac{dy}{dx} = -\sin(t).(2x + 10) = -\sin(x^2 + 10x + 12).(2x + 10)$$

Example:

$$Find\ \frac{dy}{dx}\ if\ y = \sqrt{t + 1}\ and \quad t = \frac{2}{x^2}.$$

Solution:

$$t = \frac{2}{x^2} \rightarrow t = 2x^{-2}$$

$$\frac{dt}{dx} = -4x^{-3}$$

$$\frac{dy}{dt} = \frac{1}{2}(t+1)^{-\frac{1}{2}} = \frac{1}{2\sqrt{t+1}}$$

$$\frac{dy}{dx} = \frac{dy}{dt}\ \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{t+1}}\ \frac{-4}{x^3} = \frac{1}{\sqrt{\frac{2}{x^2}+1}}\ \frac{-4}{x^3}.$$

Example:

$$Find\ \frac{dy}{dt}\ if\ y = x^2 and\ x = t^3.$$

Solution:

$$\frac{dy}{dt} = \frac{dy}{dx}\ \frac{dx}{dt}$$

$$\frac{dy}{dt} = 2x\ \ 3t^2 = 2(t^3)3t^2 = 6t^5$$

Example:

$$Find\ \frac{dy}{dt}\ if\ y + 4x^2 = 7\ and\ \ x + \frac{5}{4}t \quad = 3.$$

Solution:

$$\frac{dy}{dt} = \frac{dy}{dx}\ \frac{dx}{dt}$$

$$y = -4x^2 + 7 \quad and\ x = 3 - \frac{5}{4}t \quad .$$

$$\frac{dy}{dt} = -8\text{x}\,.\left(-\frac{5}{4}\right) = 10x.$$

$$\frac{dy}{dt} = 10 \ \left(3 - \frac{5}{4}t\right).$$

Exercise:

$$1 - Find\ \frac{dy}{dt}\ if\ \ 2x - 3y = 9\ \ and\ 2x + \frac{1}{3}t \quad = 10.$$

$$2 - Find\ \frac{dy}{dt}\ if\ \ y = \frac{x^2}{x^2 + 1}\ \ and\ x = \sqrt{2t + 1}.$$

$$3 - Find\ \frac{dz}{dt}\ if\ \ z = w^3 - w^{-1}\ \ and\ w = 3t.$$

$$4 - Find\ \frac{dr}{dt}\ if\ \ r = (s + 1)^{2/3}\ \ and\ s = 16t^2 - 4t + 10.$$

$$5 - Find\ \frac{du}{dt}\ if\ \ u = \frac{t + 1}{t}\ \ and\ t = 1 - \frac{1}{u}.$$

The exponential function:

The function $y = e^x$ is called the exponential function where is exponential number $(e = 2.71828)$ and x be any real number

$D_f = R \quad or \quad -\infty < x < \infty$

$R_f = (0, \infty) \ \ or \ \ \{y: y > 0\}$

Property of exponential function

$1 - e^x e^y = e^{x+y}$

$2 - e^{-x} = \dfrac{1}{e^x}$

$3 - \dfrac{e^x}{e^y} = e^{x-y}$

$4 - e^0 = 1$

$5 - \lim\limits_{x\to\infty} e^x = \infty$

$6 - \lim\limits_{x\to-\infty} e^x = \lim\limits_{x\to\infty} e^{-x} = \dfrac{1}{\infty} = 0.$

$7 - e^{rx} = (e^x)^r$

Derivative of exponential function

$if \ y = e^u \quad then \quad \dfrac{dy}{dx} = e^u \ u'$

Example:

Find $\frac{dy}{dx}$ of the function $y = e^{-3x}$

Solution:

$$\frac{dy}{dx} = -3e^{-3x} \; .$$

Example:

Find $\frac{dy}{dx}$ of the function $y = e^{sinx}$

Solution:

$$\frac{dy}{dx} = cosx \;\; e^{sinx} \quad .$$

Example:

Find $\frac{dy}{dx}$ of the function $y = sin(e^{-x+1})$

Solution:

$$\frac{dy}{dx} = -cos(e^{-x+1})e^{-x+1}$$

Example:

Find $\frac{dy}{dx}$ of the function $y = \frac{1}{2}(e^{x} + e^{-x})$

Solution:

$$\frac{dy}{dx} = \frac{1}{2}(e^{x} - e^{-x})$$

Example:

Find $\frac{dy}{dx}$ of the function $y = (1 + 4x + x^{2}) \;\; e^{-3x}$

Solution:

$$\frac{dy}{dx} = -3(1 + 4x + x^2)e^{-3x} + (4 + 2x)e^{-3x}$$

Remark:

$y = e^x$is continuous.(because it is differentiable).

The natural logarithm.

The function $y = log_e x$ $(e \approx 2.7183)$ is called natural logarithm with base e, we write simply as $\ln(x)$ and real lin(x)

$D_f = \{x: x > 0\}$

$R_f = -\infty < y < \infty$.

Dr. Thamer AL-khafaji

Properties:

1- $\ln\ (1) = 0$, $\quad log_e(x) = \ln(x)$
2- $\ln(a.b) = \ln(a) + \ln(b)$
3- $ln\frac{a}{b} = \ln(a) - \ln(b)$
4- $lna^r = rln(a)$
5- $ln\frac{1}{c} = \ln(1) - \ln(c) = 0 - \ln(c) = -\ln(c)$
6- $\lim_{x\to\infty} \ln(x) = \infty$
7- $\lim_{x\to 0^+} \ln(x) = -\infty$

$8 - lne^x = x$

$9 - e^{lnx} = x$

Derivative of logarithmic functions.

The derivative of logarithmic functions is:

$$if\ y = \ln(u) \rightarrow \frac{dy}{dx} = \frac{1}{u}\frac{du}{dx}.$$

Remark:

$y = \ln(x)$ is continuous.(because it is differentiable).

Example:

$$Find\ \frac{dy}{dx}, y = \ln(\text{kx}).$$

Solution:

$$\frac{dy}{dx} = \frac{k}{kx} = \frac{1}{x}.$$

Example:

$$Find\ \frac{dy}{dx}, \qquad y = \ln(\text{tanx} + \text{secx}).$$

Solution:

$$\frac{dy}{dx} = \frac{\text{sec}^2\text{x} + \text{tanx secx}}{\text{tanx} + \text{secx}} = \frac{\text{Secx (secx + tanx)}}{\text{tanx} + \text{secx}} = \text{secx}.$$

Example:

$$Find\ \frac{dy}{dx},\ y = \ln(\ln(x^2 + 3x + 1)).$$

Solution:

$$\frac{dy}{dx} = \ln(x^2 + 3x + 1) = \frac{2x + 3}{x^2 + 3x + 1}$$

$$\frac{dy}{dx} = \frac{\frac{2x+3}{x^2+3x+1}}{\ln(x^2 + 3x + 1)} = \frac{2x + 3}{(x^2 + 3x + 1)\ln(x^2 + 3x + 1)}.$$

Problems:

$$Find\ \frac{dy}{dx}\ of\ the\ following\ functions.$$

$1 - y = lnx - \frac{1}{2}\ln(1 + x^2) - \frac{\frac{1}{tanx}}{x}.$

$2 - y = \sqrt{cosx}\ \ e^{cotx}$

$3 - y = 2ln\ secx - e^{-csc^2 x}$

Equation involving ln(x) and e^x

Since $y = e^x$ and $y = \ln(x)$ are inverses of are another, we have

$e^{lnx} = x.\ \ for\ all\ x > 0.$

$lne^x = x.\ \ for\ all\ value\ of\ x.$

Example:

1-$e^{ln10} = 10.$

2-$e^{ln-2} = -2$

3- $lne^{sin10x} = sin10x$

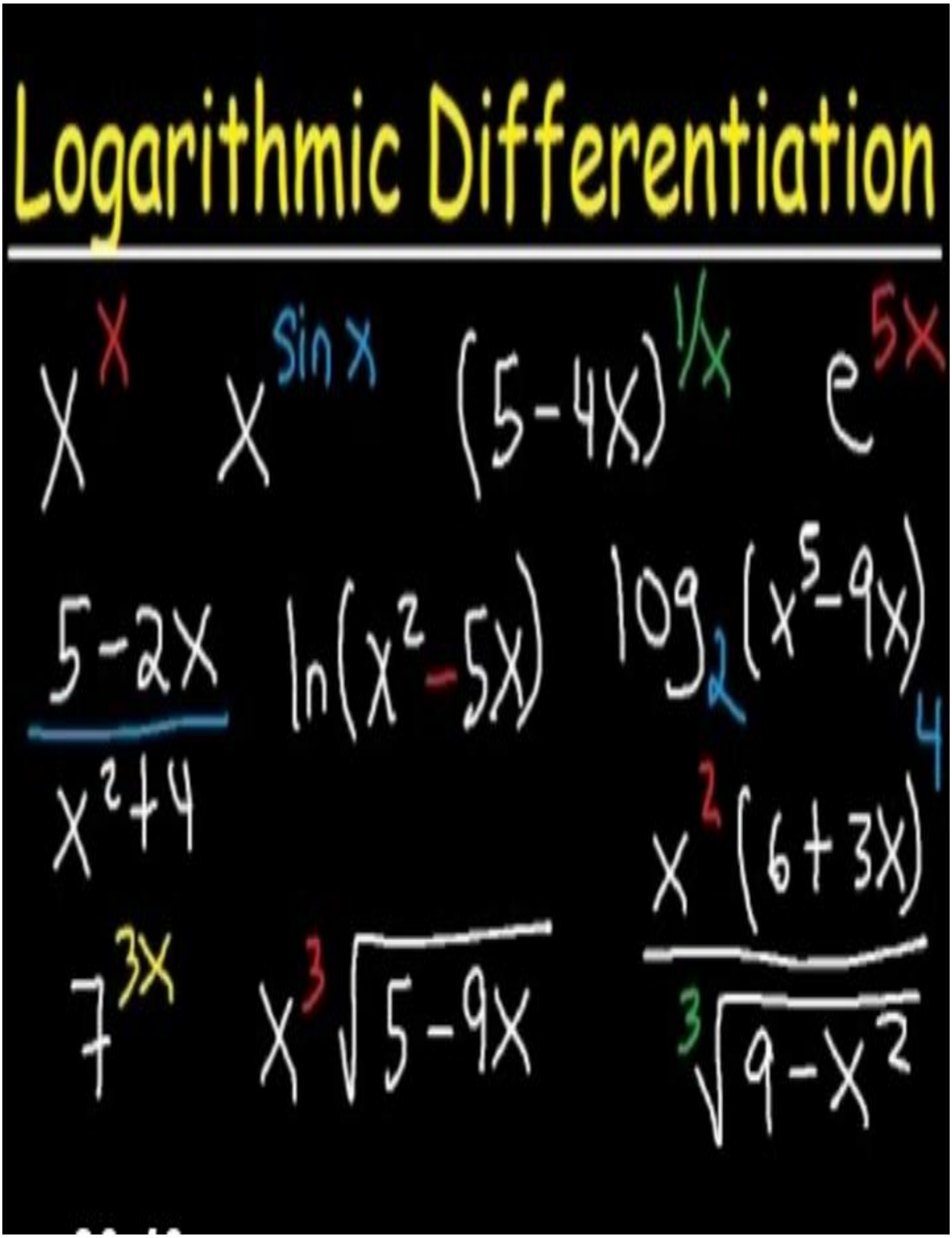
4-$ln\frac{e^{x^2}}{5cosx} = lne^{x^2} - \ln(5cosx) = x^2 - \ln(5cosx).$

Problems:

$Find\ \frac{dy}{dx}\ of\ the\ following\ functions.$

$1 - y = \sqrt{cosx}\ e^{lnsecx}$

$2 - y = lne^{tan^2}x$


Logarithmic Differentiation
x^x
$x^{\sin x}$
$(5-4x)^{1/x}$
e^{5x}
$\frac{5-2x}{x^2+4}$
$\ln(x^2-5x)$
$\log_2(x^5-9x)$
7^{3x}
$x^3\sqrt{5-9x}$
$\frac{x^2(6+3x)^4}{\sqrt[3]{9-x^2}}$

Derivative and integrals involving logarithmic functions.462k

Definition:

Let us consider the case where $y = \frac{uv}{w}$,where u,v and w and also y are functions of x.

First take logs to the base.

$$lny = lnu + lnv - lnw$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} - \frac{1}{w}\frac{dw}{dx}$$

So to get $\frac{dy}{dx}$ by it self, we merely have to multiply across by y.

Remark:

When we do this, we put the ground function that y represents

$$\frac{dy}{dx} = \frac{uv}{w}\left[\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} - \frac{1}{w}\frac{dw}{dx}\right]$$

EXample:

Differentiable with respect to x.

$$y = \frac{x^2 sinx}{cos2x}$$

Sol

$$lny = lnx^2 + lnsinx - lncos2x$$

$$\frac{1}{y}\frac{dy}{dx} = \left(\frac{1}{x^2}2x\right) + \left(\frac{1}{sinx}cosx\right) + \left(\frac{1}{cos2x}2sin2x\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + cotx + 2tan2x$$

$$\frac{dy}{dx} = \frac{x^2 sinx}{cos2x}\left(\frac{2}{x} + cotx + 2tan2x\right)$$

<u>EXample:</u>

If $y = x^4 e^{3x} tanx \quad find\ \frac{dy}{dx}$

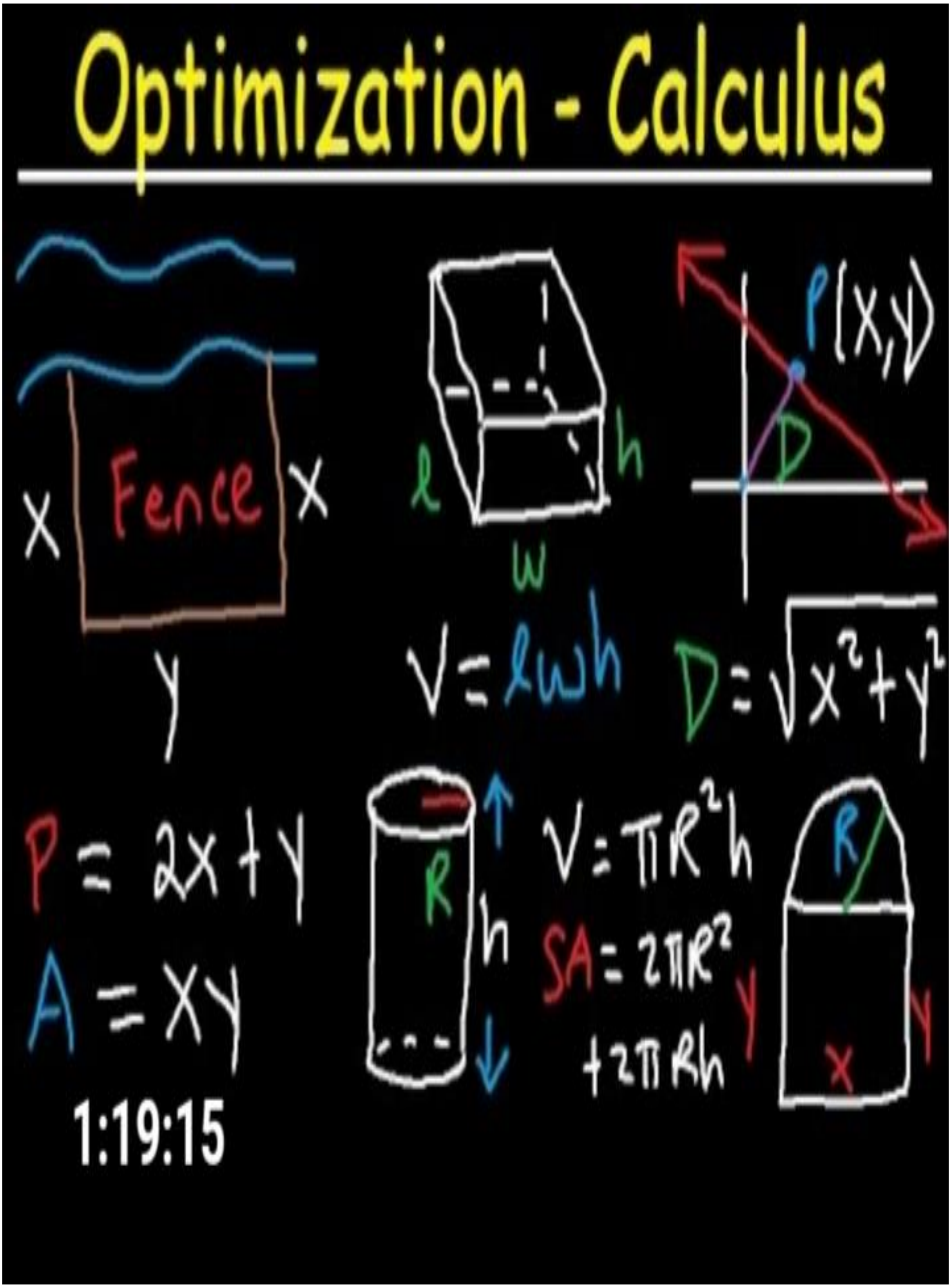
Sol

$$lny = lnx^4 + lne^{3x} - lntanx$$

$$\frac{1}{y}\frac{dy}{dx} = \left(\frac{4x^3}{x^4}\right) + \left(\frac{1}{e^{3x}}e^{3x}\right) + \left(\frac{1}{tanx}sec^2x\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \left(\frac{4}{x}\right) + (3) + \left(\frac{1}{tanx}sec^2x\right)$$

$$\frac{dy}{dx} = x^4 e^{3x} tanx\left[\left(\frac{4}{x}\right) + (3) + \left(\frac{1}{tanx}sec^2x\right)\right]$$


Optimization - Calculus
Fence
x
x
y
P = 2x + y
A = xy
1:19:15
l
h
w
V = lwh
R
h
V = πR²h
SA = 2πR²
+2πRh
P(x,y)
D
D = √x² + y²
R
y
y
x

Optimization problem.

In this chapter we will study various applications of the derivative for example we will use method of calculus to analyze functions and their graphs.

The optimization using (max, min) to solve optimize real world problems.

EXample:

Engineer has 500m of fence create rectangular factory along desert. He needs no fence along the desert it self. What are the dimensions of the factory that has the largest area.

$p = x + 2y$

$500 = x + 2y ------1$

$A = xy ---------2$

From 1 we get

$500 - 2y = x$

By 2------- $A = xy$

$A = (500 - 2y)y$

$A = 500y - 2y^2$

$A' = 500 - 4y$

$0 = 500 - 4y$

$$y = \frac{500}{4} = 125$$

$$x = 500 - 2(125)$$

$$x = 500 - 250$$

$$x = 250$$

$$A = (250m)(125m)$$

$$A = 31250m^2.$$

EXample:

Find two positive numbers whose product is 121 and whose sum is a minimum.

Sol

xy=121----------1

m=x+y-----------2

By (1)

$$y = \frac{81}{x}$$

$$m = x + \frac{121}{x}$$

$$m' = 1 - \frac{121}{x^2}$$

$$0 = 1 - \frac{121}{x^2}$$

$x^2 = \sqrt{121} \to x = 11$

$y = \frac{121}{11} = 11.$

EXample:

Find the point on the line y=3x+5 that is closet to the origin.

sol

Y=mx+b

Y=3x+5----------1

$x^2 + y^2 = d^2$---------2

$d = \sqrt{x^2 + y^2}$

$d = \sqrt{x^2 + (3x+5)^2}$

$d' = \frac{1}{2}(x^2 + (3x+5)^2)^{-\frac{1}{2}}.(2x + 2(3x+5))$

$d' = \frac{(x + 9x + 15)}{\sqrt{x^2 + (3x+5)^2}}$

10x+15=0

$x = -\frac{3}{2}$

Y=3x+5

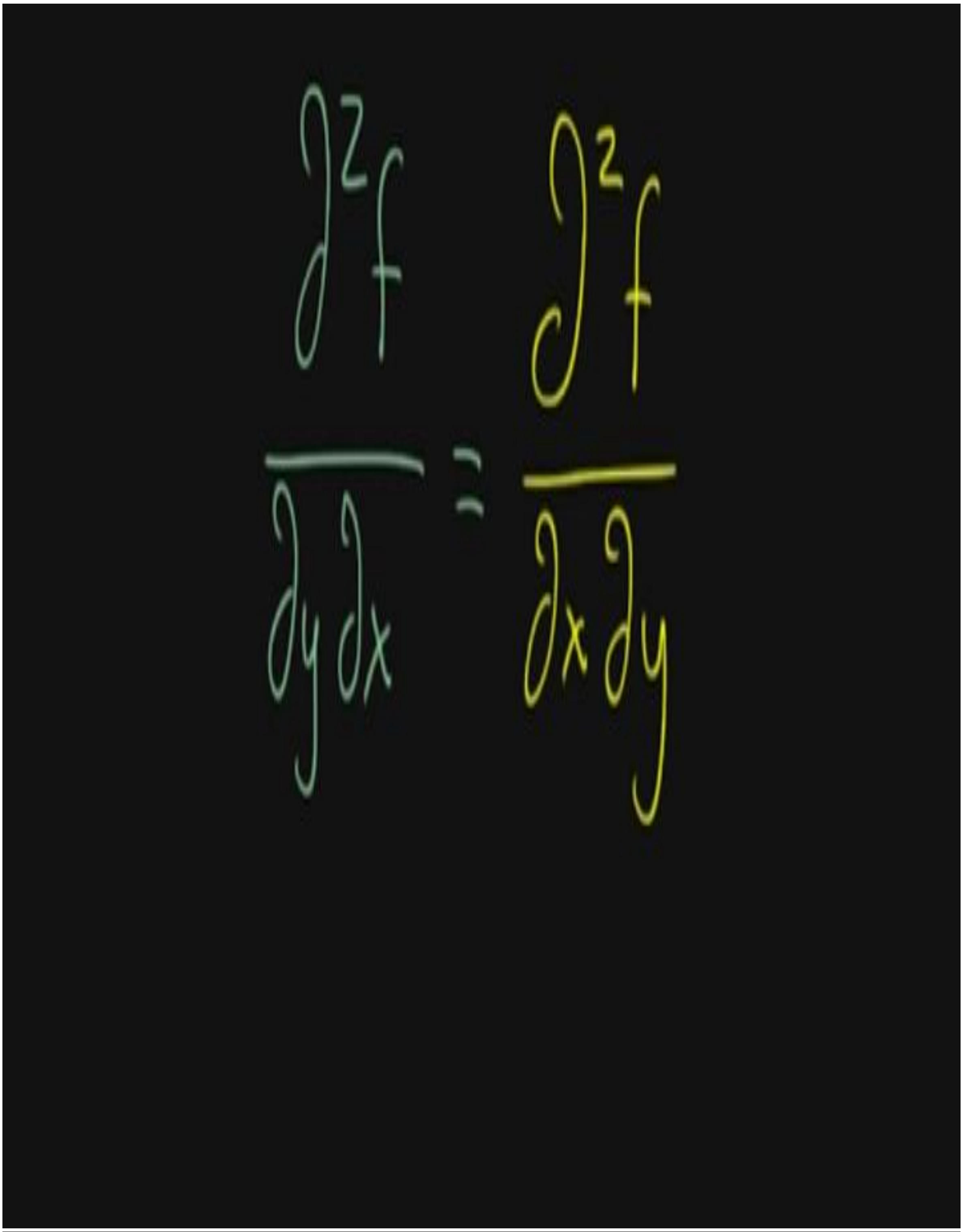
$y = 3\frac{-3}{2} + 5$

$$y = \frac{1}{2}$$

Point $= (-\frac{3}{2}, \frac{1}{2})$

Dr. Thamer AL-khafaji

Parametric Rule



Definition:

$if\ y = f(t) \quad and \quad x = g(t)\ then$

$$1 - \ y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dt} \cdot \frac{dt}{dx}.$$

$$2 - \ y'' = \frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{dy'}{dt} \cdot \frac{dt}{dx}.$$

$$3 - \ y''' = \frac{d^3y}{dx^3} = \frac{dy''}{dx} = \frac{dy''}{dt} \cdot \frac{dt}{dx}.$$

Example:

$Find\ \frac{dy}{dx}\ if\ \ y = sint \quad and \quad x = t^2 + 5t + 4.$

Solution:

$$\frac{dy}{dt} = cost \quad , \qquad \frac{dx}{dt} = 2t + 5.$$

$$then\ \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dt} \cdot \frac{dt}{dx}.$$

$$\frac{dy}{dx} = \frac{cost}{2t + 5}$$

Example:

$$Find\ \frac{d^2y}{dx^2} \quad if\ \ y = (t^2+1)^4 \ \ and \ \ x = t^4 + 6t + 1.$$

Solution:

$$\frac{dy}{dt} = 4(t^2+1)^3 2t \quad , \qquad \frac{dx}{dt} = 4t^3 + 6.$$

$$\frac{dy}{dt} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4(t^2+1)^3 2t}{4t^3+6}.$$

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{4(t^2+1)^3 2t}{4t^3+6}\right)}{4t^3+6} = \ Exc.$$

Example:

$$Find\ \frac{d^2y}{dx^2} \quad if\ \ x = \ t - t^2 \ \ and \ \ y = t - t^3.$$

Solution:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1-3t^3}{1-2t}.$$

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{1-3t^3}{1-2t}\right)}{1-2t}.$$

$$= \frac{\frac{(1-2t)(-6t)-(1-3t^3)(-2)}{(1-2t)^2}}{1-2t}$$

$$= \frac{2-6t+6t^3}{(1-2t)^3}.$$

Exercise:

$1 - Find\ \frac{dy}{dx}\ if\ \ x = 4t - 5\ \ and\ y = t^2\ when\ t = 2\ .$

$2 - Find\ \frac{dy}{dx}\ if\ \ x = 2t^2 + 2\ and\ y = 4t^2 - 1$

$when\ t = 1\ .$

$3 - Find\ \frac{dy}{dx}\ if\ \ x = \frac{t}{1-t}\ \ and\ y = t^2\ when\ t = 2\ .$

$4 - Find\ \frac{dy}{dx}\ if\ \ x = t^2\ \ and\ y = t^3\ when\ t = 2\ .$

Remark:

$$if\ x = f(x) then\ \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

Example:

$$Find\ \frac{d\ \ y}{dx}\quad if\ \ x = \sqrt{y}.$$

Solution:

$$\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$$

$$\frac{dy}{dx} = 2\sqrt{y} = 2x.$$

Example:

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Inverse trigonometric functions.ممطلوب

In this section we study inverse trigonometric functions on the one hand only, domain and Rang.

Example:

Find the domain and Rang of $\sin^{-1}(x)$.

Solution:

$f(x) = sin^{-1}(x)$ is one to one on $\frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$

$f(x) = sin^{-1}(x) \leftrightarrow x = siny$

$D_f = -1 \leq x \leq 1.$

$$R_f = \frac{-\pi}{2} \leq y \leq \frac{\pi}{2}.$$

$sin^{-1}(0) = 0$

$$sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$

$$sin^{-1}\left(\sqrt{\frac{3}{2}}\right) = \frac{\pi}{3}.$$

Example:

Find the domain and Rang of $cos^{-1}(x)$.

Solution:

$f(x) = cos^{-1}(x)$ is one to one on$[0, \pi]$

$f(x) = cos^{-1}(x) \leftrightarrow x = cosy$

$D_f = -1 \leq x \leq 1.$

$R_f = 0 \leq y \leq \pi.$

Example:

Find the domain and Rang of $tan^{-1}(x)$.

Solution:

$f(x) = tan^{-1}(x)$ is one to one on $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$f(x) = tan^{-1}(x) \leftrightarrow x = tany$

$D_f = -\infty \leq x \leq \infty.$

$$R_f = \frac{-\pi}{2} \leq y \leq \frac{\pi}{2}.$$

Example:

Find the domain and Rang of $cot^{-1}(x)$.

Solution:

$f(x) = cot^{-1}(x)$ is one to one on $(0, \pi)$

$f(x) = tan^{-1}(x) \leftrightarrow x = coty$

$D_f = R \ or \ -\infty \leq x \leq \infty$.

$R_f = (0, \pi)$.

Example:

Find the domain and Rang of $sec^{-1}(x)$.

Solution:

$f(x) = sec^{-1}(x)$ is one to one on$[0, \pi]$

$f(x) = sec^{-1}(x) \leftrightarrow x = secy$

$D_f = \{x : x \geq 1\} \cup \{x : x \leq -1\}$

$R_f = \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] / \frac{\pi}{2}$.

Example:

Find the domain and Rang of $csc^{-1}(x)$.

Solution:

$f(x) = csc^{-1}(x)$ is one to one on$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] / \{0\}$.

$f(x) = csc^{-1}(x) \leftrightarrow x = cscy$

$D_f = \{x : x \geq 1\} \cup \{x : x \leq -1\}$

$R_f = \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] / \{0\}.$

Derivative and integrals involving inverese Trigonometric functions

Thorem:

$$1 - \frac{d}{du} sin^{-1}u = \frac{du}{\sqrt{1-u^2}} \qquad -1 < u < 1.$$

$$2 - \frac{d}{du} cos^{-1}u = \frac{-du}{\sqrt{1-u^2}} \qquad -1 < u < 1.$$

$$3 - \frac{d}{du} tan^{-1}u = \frac{du}{1+u^2}$$

$$4 - \frac{d}{du} cot^{-1}u = \frac{-du}{1+u^2}$$

$$5 - \frac{d}{du} sec^{-1}u = \frac{du}{|u|\sqrt{u^2-1}} \qquad |u| > 1.$$

$$6 - \frac{d}{du} csc^{-1}u = \frac{-du}{|u|\sqrt{u^2-1}} \qquad |u| > 1.$$

Example:

$if \;\; y = sin^{-1}(x) \quad find \; \frac{dy}{dx}$

Solution.

By $\frac{d}{du} sin^{-1}u = \frac{du}{\sqrt{1-u^2}} \qquad -1 < u < 1.$

$\frac{dy}{dx} sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

Example:

$if \quad y = sin^{-1}(x^3) \quad find \ \frac{dy}{dx}$

Solution.

By $\frac{d}{du} sin^{-1}u = \frac{du}{\sqrt{1-u^2}} \qquad -1 < u < 1.$

$\frac{dy}{dx} sin^{-1}x^3 = \frac{3x^2}{\sqrt{1-(x^3)^2}}$

Example:

$\boldsymbol{y} = csc^{-1}\left(x^3 - \frac{1}{x}\right) \quad find \ \frac{dy}{dx}$

Solution.

$$By \ \frac{d}{du} csc^{-1}u = \frac{-du}{|u|\sqrt{u^2-1}} \qquad |u| > 1.$$

$$\frac{dy}{dx} csc^{-1}\left(x^3 - \frac{1}{x}\right) = \frac{-\left(3x^2 + \frac{1}{x^2}\right)}{\left|x^3 - \frac{1}{x}\right| \sqrt{\left(x^3 - \frac{1}{x}\right)^2 - 1}}$$

Example:

$\boldsymbol{y = cos^{-1}(tanx) \quad find \ \frac{dy}{dx}}$

Solution:

By: $\frac{d}{du} cos^{-1}u = \frac{-du}{\sqrt{1-u^2}} \qquad -1 < u < 1.$

$$\frac{dy}{dx} cos^{-1}(tanx) = \frac{-sec^2x}{\sqrt{1 - tan^2x}}$$

Example:

$find\ \frac{dy}{dx}$ if $y = cot^{-1}\left(\frac{1}{x}\right)$

Solution.

$$\frac{d}{du}cot^{-1}u = \frac{-du}{1+u^2}$$

$$\frac{dy}{dx}cot^{-1}\left(\frac{1}{x}\right) = \frac{-\left(\frac{1}{x^2}\right)}{1+\left(\frac{1}{x}\right)^2}$$

Exc.

$\boldsymbol{find\ \frac{dy}{dx}}$ **if** $\boldsymbol{y = \left[cos^{-1}(2x^2+3)\right]^3}$

Example:

$find\ \frac{dy}{dx}$ if $y = tan^{-1}e^{x^2}$

Solution:

By: $\frac{d}{du}tan^{-1}u = \frac{du}{1+u^2}$

$$\frac{dy}{dx}tan^{-1}e^{x^2} = \frac{2x\ e^{x^2}}{1+\left(e^{x^2}\right)^2}$$

Exc

$find\ \frac{dy}{dx}$ if $y = tan^{-1}(sinx + \cos x)$

Example:

$find\ \frac{dy}{dx}$ if $y = sec^{-1}(Lnx)$

Solution:

$$By: \frac{d}{du} sec^{-1}u = \frac{du}{|u|\sqrt{u^2 - 1}} \qquad |u| > 1.$$

$$\frac{dy}{dx} cot^{-1}(Lnx) = \frac{\frac{1}{x}}{|Lnx|\sqrt{(Lnx)^2 - 1}}$$

Example:

$find\ \frac{dy}{dx}$ if $y = cot^{-1}(x^2cosx)$

Solution:

By: $\frac{d}{du} cot^{-1}u = \frac{-du}{1+u^2}$

$$\frac{dy}{dx} cot^{-1}(x^2cosx) = \frac{-(x^2sinx + cosx.2x)}{1 + (x^2cosx)^2}$$

Exc:

$find\ \frac{dy}{dx}$ if $y = sec^{-1}(x \quad \sqrt{cosx})$

Exc

$find\ \frac{dy}{dx}$ if $y = csc^{-1}(sine^{x^3} cot \frac{2}{x^4})$

Example:

$find\ \frac{dy}{dx}$ if $y = sin(tan^{-1}u)$

Solution:

$\frac{dy}{dx}\ sin(tan^{-1}x^3) = \boldsymbol{cos}(tan^{-1}u).\frac{\boldsymbol{3x^2}}{\boldsymbol{1}+x^3}$

EXC

$1 - find\ \frac{dy}{dx}$ if $y = sec(tan^{-1}\frac{1}{x})$

$2 - find\ \frac{dy}{dx}$ if $y = cot(tan^{-1}\sqrt{x})$

$3 - find\ \frac{dy}{dx}$ if $y = tan(tan^{-1}2x)$

Inverse trigonometric functions.

We will derive some related integration formulas that involve Inverse Trigonometric functions.

Theorem.

$$1 - \int \frac{du}{\sqrt{1-u^2}} = sin^{-1}u + c$$

$$2 - \int \frac{-du}{\sqrt{1-u^2}} = cos^{-1}u + c$$

$$3 - \int \frac{du}{1+u^2} = tan^{-1}u + c$$

$$4 - \int \frac{-du}{1+u^2} = cot^{-1}u + c$$

$$5 - \int \frac{du}{u\sqrt{u^2-1}} = sec^{-1}|u| + c$$

$$6 - \int \frac{-du}{u\sqrt{u^2-1}} = csc^{-1}u + c$$

Example:

Evaluate $\int \frac{dx}{1+25x^2}$

Solution.

$$By{:}\int \frac{du}{1+u^2} = tan^{-1} u + c$$

substituting

$u = 5x, \qquad du = 5\,dx.$

yields

$$\int \frac{dx}{1+25x^2} = \frac{1}{5}\int \frac{5dx}{1+(5x)^2} = \frac{1}{5} tan^{-1}(5x) + c$$

Example:

Evaluate $\int \frac{dy}{y\sqrt{49y^2-1}}$

Solution.

$$By{:}\int \frac{du}{u\sqrt{u^2-1}} = sec^{-1}|u| + c$$

substituting

$u = \mathbf{7}y, \qquad du = \mathbf{7}\,dy.$

yields

$$\int \frac{dy}{y\sqrt{49y^2 - 1}} = \frac{1}{7}\int \frac{7dy}{y\sqrt{(7y)^2 - 1}} = \frac{1}{7}sec^{-1}(7y) + c$$

Example:

Evaluate $\int \frac{dx}{1+3x^2}$

Solution.

$$By: \int \frac{du}{1 + u^2} = tan^{-1} u + c$$

substituting

$u = \sqrt{\mathbf{3}}x, \qquad du = \sqrt{\mathbf{3}}\, dx.$

$yields$

$$\int \frac{dx}{1 + 3x^2} = \frac{\mathbf{1}}{\sqrt{\mathbf{3}}}\int \frac{\sqrt{3}dx}{1 + \left(\sqrt{\mathbf{3}}x\right)^2} = \frac{\mathbf{1}}{\sqrt{\mathbf{3}}}tan^{-1}\sqrt{\mathbf{3}}x + c$$

Example:

Evaluate $\int \frac{e^x dx}{\sqrt{1-e^{2x}}}$

Solution.

$$By \int \frac{du}{\sqrt{1 - u^2}} = sin^{-1} u + c$$

substituting

$u = \boldsymbol{e^x}, \qquad du = \boldsymbol{e^x}\, dx.$

$yields$

$$\int \frac{e^x dx}{\sqrt{1-e^{2x}}} = \int \frac{e^x dx}{\sqrt{1-(e^x)^2}} = sin^{-1}(e^x) + c.$$

EXc

$$1- \int \frac{-cosx}{\sqrt{1-sin^2 x}} dx$$

$$2- \int \frac{-1}{\sqrt{1-\left(\frac{1}{5}x\right)^2}} dx$$

$$3- \int \frac{\frac{1}{x}}{1+lnx^2} dx$$

$$4- \int \frac{-z^{\frac{-2}{3}}}{1+z^{\frac{2}{3}}} dz$$

$$5- \int \frac{x}{x^2\sqrt{x^4-1}} dx$$

$$6- \int \frac{-1}{3y\sqrt{9y^2-1}} dx$$

Integration.

Let F(x) and f(x) be two functions related as

$$\frac{\mathrm{d}}{dx}F(x) = f(x)$$

Then f(x) is called the derivative of F(x)

F(x) is called an infinite integral of f(x) and denoted by

$$F(x) = \int f(x)dx.$$

$$\int u^n \frac{du}{dx} = \frac{u^{n+1}}{n+1} + c$$

Remark:

$\frac{d}{dx}x^2 = 2x.$ $\quad \rightarrow \int 2xdx = \frac{2x^2}{2} + c = x^2 + c$

But

$$\frac{d}{dx}(x^2+3)=2x. \quad \rightarrow \int 2x dx=\frac{2x^2}{2}+c=x^2+c$$

C is called a constant of integration.

Derivative and integrals involving inverese Trigonometric functions

Integration involving Exponential

$$\int e^u\,du=e^u+c.$$

Example:

$$Find \quad \int cosx\; e^{sinx}\,dx$$

Solution.

$$\int cosx\; e^{sinx}\,dx=e^{sinx}+c.$$

Integration involving log

$$\int \frac{1}{u}du=ln|u|+c.$$

Example:

$$Find \quad \int tanx\, dx$$

Solution.

$$\int tanx\,dx = -\int \frac{-sinx}{cosx}dx = -ln|cosx| + c$$

Example:

$$Find \quad \int \frac{xe^{x^2}}{e^{x^2}}dx$$

Solution.

$$\frac{1}{2}\int \frac{2xe^{x^2}}{e^{x^2}}dx = \frac{1}{2}ln\left|e^{x^2}\right| + c$$

Integration involving Inverse Trigonometric functions

$$1 - \int \frac{du}{\sqrt{1-u^2}} = sin^{-1}u + c$$

$$2 - \int \frac{-du}{\sqrt{1-u^2}} = cos^{-1}u + c$$

$$3 - \int \frac{du}{1+u^2} = tan^{-1}u + c$$

$$4 - \int \frac{-du}{1+u^2} = cot^{-1}u + c$$

$$5 - \int \frac{du}{u\sqrt{u^2-1}} = sec^{-1}|u| + c$$

$$6-\int \frac{-du}{u\sqrt{u^2-1}} = csc^{-1}u + c$$

Ex

$$\int \frac{dx}{1+25x^2}$$

Sol

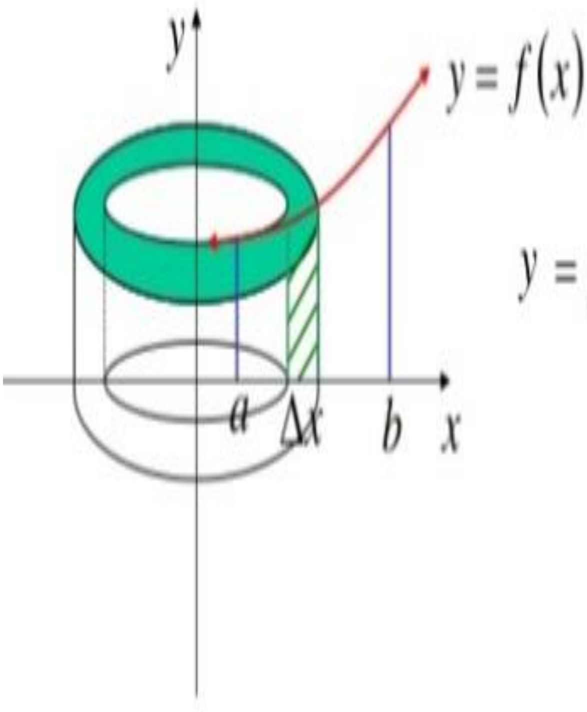
$$\int \frac{dx}{1+25x^2} = \frac{1}{5}\int \frac{5dx}{1+(5x)^2}$$
$$= \frac{1}{5}tan^{-1}(5x) + c$$

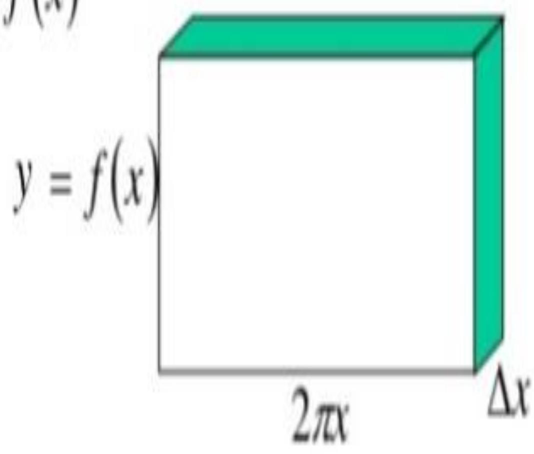
EX

$$\int \frac{dy}{y\sqrt{49y^2-1}}$$

Sol

$$\int \frac{dy}{y\sqrt{49y^2-1}} = \frac{1}{7}\int \frac{dy}{y\sqrt{(7y)^2-1}} = \frac{1}{7}sec^{-1}(7y) + c$$





$$A(x) = 2\pi x \cdot f(x)$$

$$\Delta V = 2\pi x \cdot f(x) \cdot \Delta x$$

$$V = \lim_{\Delta x \to 0} \sum_{x=a}^{b} 2\pi x \cdot f(x) \cdot \Delta x$$

$$= 2\pi \int_a^b x f(x) dx$$

Volumes by cylindrical shells.

Definition:

A cylindrical shell is a solid enclosed by two concentric right circular cylinders the volume V of a cylindrical shell with inner radius r_1, outer radius r_2 and height h can be written as

$V = [area\ of\ cross\ section].[height]$

$V = (\pi r_2^2 - \pi r_1^2)h$

$V = \pi(r_2 + r_1)(r_2 - r_1)h$

$V = 2\pi\left[\frac{1}{2}(r_2 + r_1)\right].h(r_2 - r_1)$

$\frac{1}{2}(r_2 + r_1)$ is the average radius of the shell

$r_2 - r_1$ is the thickness

$V = 2\pi$.[average radius].[height].[thickness]

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volumes by cylindrical shells About the y-Axis.

def

let f be continuous and nonnegative on [a,b], and let R be the region that is bounded above by $y = f(x)$,below by the x-axis, and on the sides by the lines $x = a \; and \; x = b$. Then the volume V of the solid of revolution that is generated by revolving the region R about the Y-axis is given by

$$v = \int_a^b 2\pi x f(x) dx$$

EX

Use the cylindrical shell to find the volume of the solid generated when the region enclosed between $y = \sqrt{x}$.

$x = 1, x = 4$, and the x-axis is revolved about the y-axis.

SOL

$$v = \int_1^4 2\pi x\sqrt{x} dx = 2\pi \int_1^4 x^{3/2} dx = \left[2\pi \frac{2}{5} x^{5/2}\right]_1^4$$

$$= \frac{4\pi}{5}[32 - 1] = \frac{124\pi}{5}$$

Example:

Use the cylindrical shell to find the volume of the solid generated when the region R in the first quadrant enclosed between is revolved about the y-axis and $x = 0, x = 1$.

SOL

$$v = \int_0^1 2\pi x(x - x^2)dx$$

$$v = 2\pi \int_0^1 (x^2 - x^3)dx$$

$$v = 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 2\pi \left[\frac{1}{3} - \frac{1}{4}\right] \quad \frac{\pi}{6}$$

<u>Maximum and Minimum values</u>

Mean value theorem

Let f be differentiable on (a,b) and continuous on [a,b],then there is at least one number c in (a,b)such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Ex

Find the value of c by used the Mean value theorem of the function $f(x) = x^2 - 6x + 4$ where $x \in [-1,7]$

Sol

1-the function is continuous

2-the function is differentiable

$f'(x) = 2x - 6$

$f'(c) = 2c - 6$

$f'(-1) = 11$

$f'(7) = 11$

$$f'(c) = \frac{f(7) - f(-1)}{8}.$$

$$2c - 6 = \frac{f(7) - f(-1)}{8}$$

$c = 3 \in (-1,7).$

Areas between two curves. 398

We showed how to find the area between two curves $y = f(x)$ and on interval on the x-axis.

If f and g are continuous functions on the interval $[a, b]$

And if $f(x) \geq g(x) forall\ x\ in\ [a, b]$,then the area of

the region bounded above by $y = f(x)$ below by $y = g(x)$, on the left by the line $x = a$, and on the right

by the line $x = b$,is

$$A = \int_b^a [f(x) - g(x)]\, dx.$$

Ex

Find the area of the region bounded above by $y = x + 6$

Bounded below by $y = x^2$and bounded on the sides by the line $x = 0$ and $x = 0$.

Sol

$$A = \int_0^2 [x + 6 - x^2]\, dx.$$

$$A = \left[\ \frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_0^2 = \frac{34}{3} - 0 = \frac{34}{3}$$

Hyperbolic functions and Hanging cables

In this section we will study certain combination of $e^x \; and \; e^{-x}$ called hyperbolic functions these functions which arise in various engineering applications have many properties in common with the trigonometric functions.

Definition:

Hyperbolic sine is → $sinhx = \frac{e^x - e^{-x}}{2}$

Hyperbolic cosine is → $coshx = \frac{e^x + e^{-x}}{2}$

Hyperbolic tangent is → $tanhx = \frac{sinhx}{coshx} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

Hyperbolic cotangent is → $cothx = \frac{coshx}{sinhx} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

Hyperbolic secant is → $sechx = \frac{1}{coshx} = \frac{2}{e^x + e^{-x}}$

Hyperbolic cosecant is → $cschx = \frac{1}{sinhx} = \frac{2}{e^x - e^{-x}}$

Example:

Find sinh x if x=0.

Sol

$$sinh0 = \frac{e^0 - e^{-0}}{2} = \frac{1-1}{2} = 0.$$

Example:

Find cosh x if x=0.

Solution:

$$cosh0 = \frac{e^0 + e^{-0}}{2} = \frac{1+1}{2} = 1.$$

Example:

Find sinh x if x=2.

Solution:

$$sinh2 = \frac{e^2 - e^{-2}}{2} \approx 3.6269.$$

Derivative Hyperbolic.

The derivative formulas for Hyperbolic functions can be obtain by expressing.

Theorem.

$$1 - \frac{d}{dx} sinhx = coshu \frac{du}{dx}$$

$$2 - \frac{d}{dx} coshu = sinhu \frac{du}{dx}$$

$$3 - \frac{d}{dx} tanhu = sech^2 u \frac{du}{dx}$$

$$4 - \frac{d}{dx} cothu = -csch^2 u \frac{du}{dx}$$

$$5 - \frac{d}{dx} sechu = -sechu\ tanh\ u \frac{du}{dx}$$

$$6 - \frac{d}{dx} cschu = -\operatorname{csch} u \coth u \frac{du}{dx}$$

Example:

Derivative formula tanhx can be obtained by formulas for Hyperbolic functions

Solution.

$$\frac{d}{dx}tanhx = \frac{d}{dx}\frac{sinhx}{coshx}$$

$$= \frac{coshx\ \frac{d}{dx}sinhx - sinhx\frac{d}{dx}coshx}{cosh^2x}$$

$$\frac{cosh^2x\ - sinh^2x}{cosh^2x} = \frac{1}{cosh^2x} = sech^2x.$$

Example:

Derivative formula coshx^3 can be obtained by formulas for Hyperbolic functions

Solution.

$$\frac{d}{dx}\cosh x^3 = 3x^2 sinhx^3$$

Example:

Derivative formula sinhx can be obtained by expressing these functions in terms of e^x and e^{-x}.

Solution.

$$\frac{d}{dx}sinhx = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = coshx$$

Example:

Derivative formula coshx can be obtained by expressing these functions in terms of e^x and e^{-x}

Solution:

$$\frac{d}{dx}coshx = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = sinhx$$

Exercise:

Derivative formula for $cothx$ can be obtained by expressing these functions in terms of e^x and e^{-x}

Exercise:

Find the derivative formula for the functions.

$1 - \mathrm{y} = \ln \mathrm{coshhx}$

$2 - y = seche^{6x}$

$4 - \mathrm{y} = \frac{\mathrm{tanhx}}{sech60}$

Hyperbolic Integrals

We will discuss methods for integrating other kinds of integrals that involve Hyperbolic Integrals.

Theorems:

$$1 - \int sinhudu = coshu + c$$

$$2 - \int coshudu = sinhu + c$$

$$3-\int sech^2udu = tanhu + c$$

$$4-\int csch^2udu = -cothu + c$$

$$5-\int sechu\ tanhudu = -sechu + c$$

$$6-\int cschu\ cothudu = -cschu + c$$

Example:

Evaluate $\int sinh\frac{1}{2}xdx$

Solution:

$$\int sinh\frac{1}{2}xdx = 2cosh\frac{1}{2}x + c$$

Example:

Evaluate= $\int cosh2xdx$

Solution.

$$\int cos2xdx = \frac{1}{2}sinh2x + c.$$